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A SIMPLE ERROR ESTIMATOR IN THE FINITE ELEMENT METHOD

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SUMMARY

Recently, a new class of error indicators and estimators for the finite element method has been introduced which is particularly easy to implement into existing finite element codes. This paper proves that the new indicators are equivalent to those analysed earlier by Babuska, thus showing that the rigorous mathematical results for the well-known jump indicator apply also for the new ones.

INTRODUCTION

Without assessment of the reliability of the results it is hardly reasonable to use numerical methods like the finite element method in such safety-sensitive areas as shape optimization of engine parts, construction of airplanes or dimensioning of nuclear power plants. Therefore, the *a posteriori* error estimation is now considered to be nearly as important as the finite element analysis itself. Much work has been done during the past years in this field (see, for example, Reference 1 for a survey) and reliable error indicators and estimators are now available.

An *error estimator* η gives a measure of the magnitude of the error in a certain, specified norm, for example the energy norm. As the error estimator should be reliable, it is highly desirable that the following property can be proven mathematically:

There exist positive constants C_1 and C_2, which are independent of the exact solution u and the specific finite element mesh, so that

$$C_1 \eta \leqslant \| e \| \leqslant C_2 \eta \tag{1}$$

holds, where $\| e \|$ is the norm of the error $e = u - U$, U being the finite element approximation to u. C_1 and C_2 should be close to 1 and the efficiency index $\eta / \| e \|$ should tend to 1 as the error goes to 0.

It has been shown[2] that property (1) holds for linear, elliptic boundary-value problems of second-order for an error estimator $\eta^{(j)}$ which is defined as

$$\eta^{(j)^2} = \sum_{i=1}^{N} \lambda_i^{(j)^2} \tag{2}$$

with *error indicators* $\lambda_i^{(j)^2}$ being summed over all elements $i = 1, \ldots, N$ and each $\lambda_i^{(j)}$ being the local projection of the error to a patch of neighbouring elements of element i. In the case of linear elements, $\lambda_i^{(j)}$ can be computed from the jumps of the derivatives of the finite element approximation across element boundaries.

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To be specific, let

$$-k\,\Delta u = f \qquad \text{on } \Omega \text{ in } R^2 \\ + \text{boundary conditions} \tag{3}$$

be the model boundary-value problem and let us approximize (3) by linear elements. Then $\lambda_i^{(j)}$ is given by

$$\lambda_i^{(j)^2} = \frac{h}{24k}\int_{\Gamma_i} J(V_x)^2 + J(V_y)^2\,\mathrm{d}\Gamma \tag{4}$$

where Γ_i is the boundary of the element Ω_i and $J(V_x)$, $J(V_y)$ are the jumps of the 'velocities'

$$V = (V_x, V_y)^{\mathrm{T}} = -k\,\mathrm{grad}\,(U) \tag{5}$$

of the finite element approximation across the edges of Ω_i. h is the diameter of the element. The reliability of (4) has not only been established mathematically by proving (1), but also been shown numerically in various test examples.[3]

Yet, it is not straightforward to implement (4) into *existing* finite element codes, because the usual data structure provides no information about the *neighbours* of a specific element.

A NEW ERROR INDICATOR

Recently, a class of new error indicators and estimators has been presented,[4] which is easy to implement into existing finite element codes without adding a new data structure to the programs. Consider again model problem (3) and let

$$v = (v_x, v_y)^{\mathrm{T}} = -\,k\,\mathrm{grad}\,(u) \tag{6}$$

be the velocity of the exact solution of (3). The error in energy norm over an element Ω_i is then given by

$$\|e\|^2_{E,\Omega_i} = -\int_{\Omega_i}(v-V)^{\mathrm{T}}\mathrm{grad}(u-U)\,\mathrm{d}\Omega = \int_{\Omega_i}(v-V)^{\mathrm{T}}\begin{pmatrix}1/k & 0\\ 0 & 1/k\end{pmatrix}(v-V)\,\mathrm{d}\Omega \tag{7}$$

The finite element approximation V to the velocity v is discontinuous across element edges and nearly all commercial codes provide means to compute *smoothed velocities* (resp. smoothed stresses in elasticity problems) by various smoothing techniques, e.g. nodal averaging, local or global smoothing.[5,6]

Let us now denote these smoothed velocities by $\bar{V} = (\bar{V}_x, \bar{V}_y)^{\mathrm{T}}$ and replace v by $\bar{V}$ in expression (7) for the error in energy norm to define

$$\lambda_i^{(s)^2} = \int_{\Omega_i}(\bar{V} - V)^{\mathrm{T}}\begin{pmatrix}1/k & 0\\ 0 & 1/k\end{pmatrix}(\bar{V} - V)\,\mathrm{d}\Omega \tag{8}$$

If we assume that $\bar{V}$ is a better approximation to v than V, then $\lambda_i^{(s)}$, which will be called *'smoothed-stress' error indicator* is an approximation to the error in energy norm in the element i. $\lambda_i^{(s)}$ is particularly easy to compute, as $\bar{V}$ is usually available and the integral can be evaluated numerically using the element mass matrices for (3).

This derivation of the smoothed-stress indicators is, of course, heuristic and assumes implicitly superconvergence of the smoothed velocities resp. stresses. The reliability of the estimators has been shown in various test examples.[4] In addition, we will give in the following a mathematical foundation of these indicators and corresponding estimators in showing that (1) holds, too. We will do this, proving that $\lambda_i^{(s)}$ is equivalent to $\lambda_i^{(j)}$, i.e. that there are constants C_3 and C_4, so that

$$C_3\,\lambda_i^{(j)} \leqslant \lambda_i^{(s)} \leqslant C_4\,\lambda_i^{(j)} \qquad C_3, C_4 > 0 \tag{9}$$

holds.

A ONE-DIMENSIONAL PROBLEM

Consider first the one-dimensional problem

$$-k\,u''(x) = f(x) \qquad x \in (a,b) \tag{10}$$

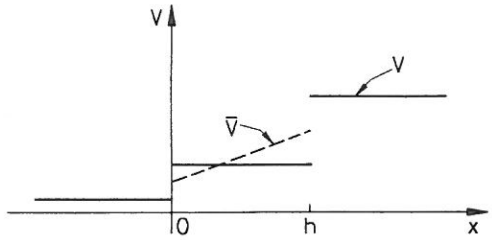

Figure 1.

with appropriate boundary conditions and a finite element approximation to u with linear elements:

$$U(x) = \sum_{i=1}^{n} N_i(x)\, U_i \tag{11}$$

The 'jump'-error indicator and estimator has the following form:

$$\lambda_i^{(j)^2} = \frac{hk}{24} (J_1(U')^2 + J_r(U')^2)$$

$$\eta^{(j)^2} = \sum_{i=1}^{n} \lambda_i^{(j)^2} \tag{12}$$

$$\mathrm{h} = \mathrm{x}_{i+1} - x_i$$

where $J_1(U')$ and $J_r(U')$ are the jumps of U' on the left and right node, respectively, of element i.

The smoothed stress indicator resp. estimator is given by

$$\lambda_i^{(s)^2} = \int_{x_i}^{x_{i+1}} (V - \bar{V})\, k^{-1} (V - \bar{V})\, \mathrm{d}x$$

$$\eta^{(s)^2} = \sum_{i=1}^{n} \lambda_i^{(s)^2} \tag{13}$$

where V is the 'velocity' or 'stress' of the finite element approximation:

$$V = -\,k\, U' \tag{14}$$

$\bar{\mathrm{V}}$ is a smoothed stress obtained from the finite element approximation V. V has a jump of $k\, J_1(U')$ and $k\, J_r(U')$ at the left and right node of an element. Assume now that $\bar{V}$ is obtained by linearly interpolating the averaged nodal values of V (see Figure 1). Then, $\bar{V}$ can be written as

$$\bar{V}(x) = V(x) - 1/2\, k\, J_1(U') + 1/(2h)\, k\, (J_1(U') + J_r(U'))\, x \tag{15}$$

Inserting (15) into (13) we get

$$\lambda_i^{(s)^2} = k/4 \int_0^h \left((J_1 + J_r) \frac{x}{h} - J_1 \right)^2 \mathrm{d}x = \frac{kh}{12} (J_1^2 - J_1 J_r + J_r^2) \tag{16}$$

Furthermore, we have

$$0{\cdot}5\, (J_1^2 + J_r^2) \leq (J_1^2 - J_1 J_r + J_r^2) \leq 1{\cdot}5\, (J_1^2 + J_r^2) \tag{17}$$

Using (17), (16) and (12) we get immediately

$$\lambda_i^{(j)^2} \leq \lambda_i^{(s)^2} \leq 3\, \lambda_i^{(j)^2} \tag{18}$$

The extremal cases in (18) are the following:

(a) $J_1 = J_r$ (see Figure 2). In this case $\lambda_i^{(j)} = \lambda_i^{(s)}$, i.e. the smoothed-stress indicator, is equal to the jump indicator.

(b) $J_1 = -\,J_r$ (see Figure 3). In this case we have $\lambda_i^{(s)} = \sqrt{3}\, \lambda_i^{(j)}$.

Case (b) will never occur in a properly refined mesh. If the mesh is refined such that the error indicators are distributed equally (what is attempted in an adaptive mesh-refinement), the limiting

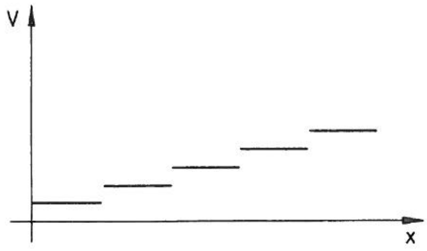

Figure 2.

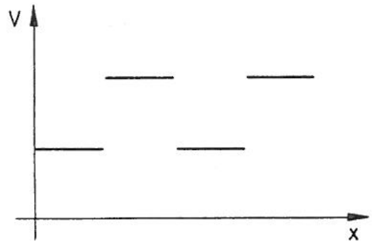

Figure 3.

case will always be (a). This means that the corresponding error estimators $\eta^{(s)}$ and $\eta^{(j)}$ will be asymptotically the same.

THE TWO-DIMENSIONAL PROBLEM

Consider again model problem (3), which shall be approximated by bilinear finite elements. For simplicity, we will first assume a *square* mesh with side-length h of each element. Jump indicators and smoothed-stress indicators are defined in equations (4) and (8). Assume now that the smoothed velocities $\bar{V}_x$ and $\bar{V}_y$ are defined as

$$\bar{V}_x(x,y) = \sum_{i=1}^{4} N_i(x,y)\, \bar{V}_{x,i}$$

$$\bar{V}_y(x,y) = \sum_{i=1}^{4} N_i(x,y)\, \bar{V}_{y,i} \tag{19}$$

where N_i are the bilinear shape functions and $\bar{V}_{x,i}$, $\bar{V}_{y,i}$ are the averaged nodal velocities.

Consider now Figure 4. In element 1, the differences of the smoothed velocities at node 3 and the finite element velocities can be written as

$$\bar{V}_x - V_x = (3\,J^{(d)}(V_x) + 3\,J^{(1)}(V_x) + J^{(r)}(V_x) + J^{(u)}(V_x))/8 \tag{20}$$

$$\bar{V}_y - V_y = (3\,J^{(d)}(V_y) + 3\,J^{(1)}(V_y) + J^{(r)}(V_y) + J^{(u)}(V_y))/8 \tag{21}$$

From the continuity of the finite element approximation we get the following relations:

$$J^{(1)}(V_x) = J^{(r)}(V_x) = J^{(d)}(V_y) = J^{(u)}(V_y) = 0 \tag{22}$$

$$J^{(1)}(V_y) = J^{(r)}(V_y) \tag{23}$$

$$J^{(d)}(V_x) = J^{(u)}(V_x) \tag{24}$$

Inserting (22)–(24) into (20) and (21) yields

$$\bar{V}_x - V_x = 0{\cdot}5\,J^{(d)}(V_x) \tag{25}$$

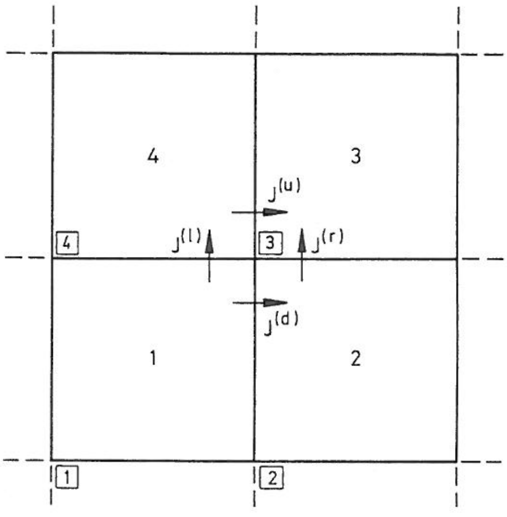

Figure 4.

and

$$\bar{V}_y - V_y = 0{\cdot}5\,J^{(1)}(V_y) \tag{26}$$

As the difference of the smoothed and the original approximation is now reduced to only one jump at each node, we can omit the superscript for the jumps and denote the jumps of V_x and V_y at node i, $i = 1,\ldots,4$ by $J_{i,x}$ and $J_{i,y}$. Integral (8) can be considered a 'mass' of element i, so it can be rewritten in the following form, taking into account the appropriate signs of the jumps at nodes 1 to 4:

$$\begin{aligned}\lambda_i^{(s)^2} = \frac{h^2}{16k}\,&((-J_{1,x},J_{2,x},J_{3,x},-J_{4,x})\,M\,(-J_{1,x},J_{2,x},J_{3,x},-J_{4,x})^{\mathrm{T}} \\ &+ (-J_{1,y},-J_{2,y},J_{3,y},J_{4,y})\,M\,(-J_{1,y},-J_{2,y},J_{3,y},J_{4,y})^{\mathrm{T}})\end{aligned} \tag{27}$$

Here, M is a mass matrix for a standard square element of side-length 2.

Evaluation of (27) with a consistent mass matrix (see, for example, Reference 6), which gives the exact integrals for linear jumps yields

$$\begin{aligned}\lambda_i^{(s,c)^2} = {}&\frac{h^2}{144\,k} \\ &(4\,(J_{i,x}^2 + J_{2,x}^2 + J_{3,x}^2 + J_{4,x}^2 - J_{1,x}J_{2,x} + J_{1,x}J_{4,x} + J_{2,x}J_{3,x} - J_{3,x}J_{4,x}) \\ &- 2\,(J_{1,x}J_{3,x} + J_{2,x}J_{4,x}) + \\ &+ 4\,(J_{1,y}^2 + J_{2,y}^2 + J_{3,y}^2 + J_{4,y}^2 + J_{1,y}J_{2,y} - J_{1,y}J_{4,y} - J_{2,y}J_{3,y} + J_{3,y}J_{4,y}) \\ &- 2(J_{1,y}J_{3,y} + J_{2,y}J_{4,y}))\end{aligned} \tag{28}$$

Evaluating (27) with a lumped mass matrix gives

$$\lambda_i^{(s,1)^2} = \frac{h^2}{16k}\left(\sum_{i=1}^{4} J_{i,x}^2 + J_{i,y}^2\right) \tag{29}$$

In Reference 3 the following numerical approximation of the jump indicator (4) is suggested:

$$\lambda_i^{(j)^2} = \frac{h^2}{48k}\left(\sum_{i=1}^{4} J_{i,x}^2 + J_{i,y}^2\right) \tag{30}$$

We are now able to show the equivalence of the jump indicator and the smoothed-stress indicators and thus, summing over all elements, the desired property (1) for the related error estimators.

Obviously, the indicators $\lambda_i^{(s,1)}$ in (29) of the lumped evaluation of integral (8) is $\sqrt{3}$ times $\lambda_i^{(j)}$, so (1) holds with

$$C_1' = C_1/\sqrt{3} \;;\; C_2' = C_2/\sqrt{3} \tag{31}$$

To show the equivalence of $\lambda_i^{(j)}$ and $\lambda_i^{(s,c)}$ consider first the expression

$$\mu := 4(a^2 + b^2 + c^2 + d^2 - ab + ad + bc - cd) - 2(ac + bd) \tag{32}$$

corresponding to the jumps of V_x resp. V_y in (28). From

$$0 \leqslant (a-b+d-c)^2 = a^2 + b^2 + c^2 + d^2 + 2(-ab + ad - ac - bd + bc - cd) \tag{33}$$

we get immediately

$$\begin{aligned} \mu &\geqslant 3\,(a^2 + b^2 + c^2 + d^2) + 2(ad - ab + bc - cd) = \\ &(a - b)^2 + (a + d)^2 + (b + c)^2 + (c - d)^2 + a^2 + b^2 + c^2 + d^2 \end{aligned} \tag{34}$$

and thus

$$\mu \geqslant a^2 + b^2 + c^2 + d^2 \tag{35}$$

Using now $|2ab| \leqslant a^2 + b^2$ we get

$$\mu \leqslant 9(a^2 + b^2 + c^2 + d^2) \tag{36}$$

Inserting (35) and (36) into (28) for the jumps in V_x and V_y yields the desired result:

$$\frac{h^2}{144k}\left(\sum_{i=1}^{4} J_{1,x}^2 + J_{i,y}^2\right) \leqslant \lambda_i^{(s,c)^2} \leqslant \frac{h^2}{16k}\left(\sum_{i=1}^{4} J_{i,x}^2 + J_{i,y}^2\right) \tag{37}$$

Thus, inequality (1) holds for $\eta^{(s,c)}$ with

$$C_1' = C_1/\sqrt{3} \;\;;\; C_2' = C_2\sqrt{3}$$

Finally, the equivalence of $\eta^{(j)}$ and $\eta^{(s)}$ for general bilinear elements follows from the fact that then the expressions of $\eta^{(j)}$ and $\eta^{(s)}$ differ from (28) to (30) by constants depending only on the Jacobian J of the bilinear mapping. As long as the Jacobian is bounded, i.e. there exist constants $c_1,\, c_2 \geqslant c > 0$ so that

$$c_1\, h^2 \leqslant |\det J| \leqslant c_2\, h^2 \tag{38}$$

the estimators are equivalent again, proving property (1) for the smoothed-stress estimators.

CONCLUSIONS

It has been shown that a class of recently introduced error indicators and estimators for bilinear finite element approximations to linear, elliptic boundary-value problems of second order is equivalent to estimators which were presented earlier in Reference 3. Therefore, the mathematical property of being a simultaneous upper and lower estimator in the energy norm carries over to the new estimators. The new estimators are particularly easy to implement into existing finite element codes and promise cheap and reliable quality control of finite element results. Moreover, they can easily be adapted to other norms or higher order elements, although in these cases the proof of (1) is not straightforward.

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REFERENCES

1. I. Babuska, O. C. Zienkiewicz *et al.* (eds). *Accuracy Estimation and Adaptivity for Finite Elements*, Wiley, New York, 1986.
2. I. Babuska and W. C. Rheinboldt, 'Error estimates for adaptive finite element computations', *SIAM J. Numer. Anal.*, **15**, 736–754 (1978).
3. I. Babuska and A. Miller, 'A posteriori error estimates and adaptive techniques for the finite element method', *Tech. Note BN–968*, Inst. for Phys. Sci. and Techn. Univ. of Maryland (1981).
4. O. C. Zienkiewicz and J. Z. Zhu, 'A simple error estimator and adaptive procedure for practical engineering analysis', *Int. j. numer. methods eng.*, **24**, 337–357 (1987).
5. E. Hinton and J. Campbell, 'Local and global smoothing of discontinuous finite element functions using a least square method', *Int. j. numer. methods. eng.*, **8** 461–480 (1974).
6. O. C. Zienkiewicz, *The Finite Element Method*, 3rd edn, McGraw-Hill, New York, 1977.