AN IMPLEMENTATION OF THE HP-VERSION OF THE FINITE ELEMENT METHOD FOR REISSNER-MINDLIN-PLATE PROBLEMS

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Abstract

Reissner-Mindlin-plate theory is still a topic of research in finite element analysis. One reason for the continuous development of new plate elements is that it is still difficult to construct elements which are accurate and stable against the well known shear locking effect. In this paper we suggest an approach which allows high order polynomial degrees of the shape functions for deflection and rotations. A balanced adaptive mesh-refinement and increase of the polynomial degree in an hp-version finite element program is presented and it is shown in numerical examples that the results are highly accurate and that high order elements show virtually no shear locking even for very small plate thickness.

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1. Introduction

During the past 25 years numerous plate elements have been presented in the literature. One of the main reasons for this continuous interest in improving finite element discretizations for plate and shell models is an inherent difficulty in the formulation of the problem. Kirchhoff-plate-theory is only valid for thin plates and, as it is described by an equation of fourth order, has to be approximated (if conformal elements are to be used) by $C^1$-elements. Reissner-Mindlin-plate-theory, on the other hand, is valid for moderately thick plates, too, and is described by a system of three equations of second order. The problems in the discretization of Reissner-Mindlin-plates is mainly due to the shear locking phenomenon for small plate thickness. The reason for this can already be seen in the one-dimensional analogon of a Timoshenko-beam.

Let $w(x)$ be the displacement, $\phi(x)$ the rotation at a point $x$ of a beam with thickness $t$, $x$ in the interval $(0,1)$. Then the inner energy of the beam is given by

$$\Pi = C_1 \int_0^1 \phi'^2 + \frac{C_2}{t^2} (w' - \phi)^2 \, dx \quad (1)$$

where $C_1$ and $C_2$ are material constants depending on elasticity and shear modulus. In (1) $1/t^2$ plays the role of a Lagrange multiplier imposing the constraint $w' = \phi$ as $t$ tends to zero (which is just the Kirchhoff assumption).

This implies immediately that, for thin beams, the spaces for the numerical approximation $W$ and $\phi$ to $w$ and $\phi$ have to be constructed in such a way that $W'$ can equal $\phi$ for small $t$. If only linear elements are used for $W$ and $\phi$ and if the beam is built-in, it can readily be seen that then the only solution is $W \equiv \phi \equiv 0$, the well-known locking solution.

Many finite element approaches try to overcome the locking problem by numerical techniques like reduced or selective integration \(^1\) or by 'assumed-stress'-elements \(^2\). These approaches 'simulate' different polynomial degrees for $W$ and $\phi$ by special numerical integration methods. In this paper we will try to tackle the difficulty at its root, we will allow higher order (and possibly different) polynomial degree for deflection
and rotation in order to make the approximation space 'rich' enough to be able to meet the Kirchhoff-constraint. This is done adaptively in the framework of a so-called hp-version finite element method which has recently been shown to be a very promising new approach to yield highly accurate results in elasticity and potential problems\(^3\)-\(^{10}\). The p-version for the Kirchhoff plate model and for a 3D-plate-approximation was investigated in\(^{11}\) and a plate theory based on higher order polynomial approximations over the thickness of the plate was presented in\(^{12}\).

2. The p- and hp-version of the finite element method

As the p- and especially the hp-version of the finite element method have only recently been presented, the basic ideas shall be described briefly in this chapter. The 'usual' h-version finite element method achieves convergence by successive refinement of the mesh using shape functions of low polynomial degree. The p-version, on the other hand, uses a fixed mesh and increases the polynomial degree p. The hp-version, finally, is a combination of both, i.e. refines the mesh and increases simultaneously the polynomial degree. It can be shown\(^6\) that an optimal combination of h- and p-refinement yields exponential rate of convergence even if there are singularities in the exact solution. Consider, as a model problem

\[
\begin{align*}
\Delta u &= f \\
\mathbf{u} &= \mathbf{u}_0 \\
\frac{\partial \mathbf{u}}{\partial n} &= g_0
\end{align*}
\]

on \(\Omega \subseteq \mathbb{R}^2\) on \(\Gamma_1\) on \(\partial \Omega \setminus \Gamma_1\) \hspace{1cm} (2)

Let, as a first case, the exact solution \(\mathbf{u}\) of (2) be analytic. Then the p-version converges exponentially to the exact solution, i.e.

\[
\| \mathbf{u} - \mathbf{u}_E \|_E = C e^{-N(p)^{1/2}}
\]

where \(\|\cdot\|_E\) is the energy norm, \(\mathbf{u}\) the finite element approximation, \(C\) a positive constant and \(N(p)\) the number of degrees of freedom which depends on the polynomial degree \(p\) of the approximation.

Let now \(0 < \mu_i \leq \mu_{i+1}\) for all \(i \in \mathbb{N}\), \(r\) and \(\theta\) be polar coordinates, \(g_0\) and \(u_1\) smooth functions, and \(C_i\) real constants.
If the exact solution $u$ is of the form
\[ u = u_1 + \sum_{i=1}^{\infty} c_i r^\mu_1 g_i(\theta) \] (4)
the p-version converges algebraically, i.e.
\[ \| e \|_E \leq C N(p)^{\alpha} \] (5)
where $\alpha$ depends essentially on the smallest exponent $\mu_1$. The proof of (3) and (5) can be found in \textsuperscript{6}. Note that (4) is the typical form of a solution on domains with corners.

Beyond the asymptotic behaviour the convergence of the p-version in the pre-asymptotic range is of special interest for the construction of an hp-version FEM. In figure 1a-c three finite element meshes for an L-shaped domain are shown, in figure 2 the corresponding convergence of the error in the energy norm is plotted. Each of the curves shows a typical inverted S-formed behaviour, i.e. an exponential pre-asymptotic range (bent down) and an algebraic asymptotic range (levelling off to a straight line).

It can be shown that the lower left envelope of the curves (dashed line) itself is bent down, i.e. shows exponential rate of convergence. The optimal strategy with respect to a minimization of the error for a given number of degrees of freedom is, for the example of figure 2, to use mesh 1 for polynomial degree 1, 2 and 3, to switch to the refined mesh 2 with polynomial degree 3 and 4 and to use polynomial degrees 4 and 5 on mesh 3.

Of course, the optimal combination of mesh and polynomial degree depends on the special situation at the singular point of the domain, i.e. on $c_1$ and $\mu_1$ in the expansion (4).

A detailed analysis and a prediction of the error for domains with corners can be found in \textsuperscript{8} and \textsuperscript{9} in the framework of a finite element expert system.

Another approach was first presented in \textsuperscript{10} for an adaptive hp-version for potential problems. The basic idea for the algorithm in the expert system and in the adaptive code are the same. The computation starts on a very coarse mesh where the elements are separated into two classes. The first class, (noncritical elements), are all elements where the exact solution is expected to be smooth; there no mesh refinement at all is necessary, a pure p-version converges exponentially.
The other class (critical elements) are all elements which are adjacent to reentrant corners or points of change of boundary condition. These elements have to be refined and the polynomial degree has to be increased as motivated by the convergence curves of figure 2.

The adaptive hp-version controls mesh refinement and increase of polynomial degree by a *posteriori* error estimation for the error in some norm. After each finite element solution error indicators $\lambda_i$ for each element $i$ are computed. These give an estimation for the influence of element $i$ to the error. The total error $\|e\|_{H^1}$ is estimated by an error estimator $\eta$ which can be used as a breakoff criterion for the refinement process. For the definition of error indicators and estimators, see, for example 13.

With these concepts the following adaptive hp-version algorithm can be formulated which will be used for approximation of the Reissner-Mindlin-plate-problem.
Define basic mesh, set $p=1$ and choose $0 < \tau \leq 1$.

\[ \downarrow \]

Identify critical elements.

\[ \downarrow \]

Perform a finite element computation.

\[ \downarrow \]

Compute error indicators $\lambda_i$ and an error estimator $\eta$.

\[ \downarrow \]

\[ \eta \leq \text{acceptable error} \quad \Rightarrow \quad \text{STOP} \]

\[ \downarrow \]

$n$

\[ \lambda_{\text{max}} := \max_{i=1,\ldots,n} (\lambda_i) \]

\[ \lambda_{\text{crit}} := \gamma \lambda_{\text{max}} \]

For each element $i$

\[ \downarrow \]

If $\lambda_i \geq \lambda_{\text{crit}}$ then
  
  If element is noncritical then
    increase polynomial degree by 1
  
  Else
    refine geometrically

End if

End if

An hp-version program can be derived from a p-version program by adding the component of local mesh refinement which is controlled by the algorithm defined in the previous chapter. The basic concepts of a hierarchical p-version FEM have been described for example in 4. So we can restrict here to those aspects which are especially important for the Reissner-Mindlin-problem and to a description of a data structure which is appropriate for h- and p-refinement. This will

- allow to choose the polynomial degree independently for deflection and displacements

- allow to vary the polynomial degree from element to element

- guarantee \( C^0 \)-continuity of the finite element approximation in each variable

- allow to refine the mesh geometrically at singular points of the exact solution.

We will restrict our description to quadrilateral elements yet it should be mentioned that a generalizations to elements of more complex shape is straightforward.

For the description of Reissner-Mindlin's plate problem let \( E \) be Young's modulus, \( \nu \) the Poisson ratio and \( G \) the shear modulus, \( t \) the thickness of the plate and \( K = \pi / t \) Mindlin's constant. We denote the deflection at a point \((x,y)\) in the domain \( \Omega \subset \mathbb{R}^2 \) by \( w \), the rotation about the \( y \)-axis by \( \theta_1 \) and the rotation about the \( x \)-axis by \( \theta_2 \).

The generalized strains \( \varepsilon := (\zeta^T, \phi^T)^T \) of the plate are composed of a strain vector

\[
\zeta = \left( \begin{array}{c}
\frac{\partial \theta_1}{\partial x} \\
\frac{\partial \theta_2}{\partial y} \\
- \left( \frac{\partial \theta_1}{\partial y} + \frac{\partial \theta_2}{\partial x} \right)
\end{array} \right)
\]

and a rotation vector

\[
\phi = \left( \begin{array}{c}
\frac{\partial w}{\partial x} - \theta_1 \\
\frac{\partial w}{\partial y} - \theta_2
\end{array} \right)^T
\]
The generalized stress resultants

\[ \sigma = (m_x, m_y, m_{xy}, q_x, q_y) \]  

(8)

are grouped into two parts, the bending moments \( M = (m_x, m_y, m_{xy}) \)

and the shear forces \( Q = (q_x, q_y) \) with the form

\[ M = D D_b \zeta \]

\[ Q = S D_s \phi \]

(9)

where

\[ D = \frac{E t^3}{12(1-\nu^2)} ; \quad S = \frac{G t^3}{12} \]

The material matrices have the form

\[ D_b := \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} ; \quad D_s := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} ; \quad D_M := \begin{bmatrix} D & D_b & 0 \\ 0 & S & D_s \end{bmatrix} \]

(10)

Then the inner energy of the plate is given by

\[ \Pi = \int_D \zeta^T D_b \zeta + S \phi^T D_s \phi \, d\Omega = \int_\Omega \epsilon^T D_M \epsilon \, d\Omega \]

(11)

As already mentioned in the introduction, the locking problem for thin plates will be overcome by higher (and possibly different) polynomial degree \( p \) for \( w \) and \( \theta_1 \) respectively \( \theta_2 \). The functions \( w, \theta_1 \) and \( \theta_2 \) are approximated by \( W, \phi_1 \) and \( \phi_2 \), each of which has a finite element representation of the form

\[ V = \sum_{k=1}^{n} a_k N_k \]

(12)
The shape functions $N_k$ are defined on a standard square $[-1,1] \times [-1,1]$ in the local coordinates $\xi, \eta$. They can be grouped into three classes. The first group are basic modes which are the 'usual' shape functions for bilinear elements.

The second class are edge modes. For their definition, let

$$\{ P_i(\xi) \mid i \geq 2, \xi \in [-1,1], P_i(-1) = P_i(1) = 0 \} \tag{13}$$

be a class of polynomials $P_i$ of degree $i$. A possible choice for $P_i$ is given by the integral of the Legendre polynomials defined by

$$P_i(\xi) = \int_{-1}^{\xi} L_{i-1}(t) \, dt \quad \text{with} \quad L_n(x) = \frac{1}{n!} \frac{d^n(x^2-1)^n}{dx^n} \tag{14}$$

Then the edge functions for the edge $\eta = 1$ (and analogously for the other edges of the standard element) are defined by

$$N_k(\xi, \eta) = P_i(\xi) \frac{\eta + 1}{2} \tag{15}$$

The third class are bubble modes defined by

$$N_k(\xi, \eta) = P_i(\xi) P_j(\eta), \quad i,j \geq 2 \tag{16}$$

which are identically zero on all edges of the elements.

Let now $\{ I = \Omega_i \mid i=1,...,n \}$ be the set of all (quadrilateral) elements, $\{ E = E_i \mid i=1,...,m \}$ the set of all edges of the finite element mesh.

Then for each of the fields $W$, $\phi_1$, and $\phi_2$ the following data structure defines the finite element approximation.

**Basic and bubble modes**

Let $\Omega_i$ be an element. Then

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ID_NODE1, ID_NODE2, ID_NODE3, ID_NODE4
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defines the equation numbers of the basic modes associated to node 1 to node 4 of element $\Omega_i$. 

Bubble modes are defined by triples

\((I_1, J_1, \text{ID}_{IJ1}), (I_2, J_2, \text{ID}_{IJ2}), \ldots\)

The pair of integers \(I_k^r, J_k^r\) defines the mode

\[ N_k^r = P_{I_k^r}(\xi) P_{J_k^r}(\eta) \]

with the equation number ID_{IJk}.

**Edge modes**

For each quadrilateral element a list of 4 edges

\[ E_{i_1}, E_{i_2}, E_{i_3}, E_{i_4} \]

is defined. Each of \(E_{i_1}\) to \(E_{i_4}\) is a pointer to the edge data structure. For each edge \(E_j\) a list of pairs

\((I_1, \text{ID}_{I1}), (I_2, \text{ID}_{I2}), \ldots\)

is stored.

The integer \(I_k^r\) gives the polynomial degree and ID_{Ik} the equation number of the corresponding shape function \(N_{I_k}^r\) in (15).

The element stiffness matrix for element \(\Omega_i\) can now be computed as energy product of all modes associated to \(\Omega_i\).

Edge modes, which are together with the basic modes the only shape functions with support of more than one quadrilateral, are defined consistently in elements with a common edge via the edge data structure. Thus continuity of the finite element approximation is guaranteed automatically. It is also obvious that the data structure allows to choose the polynomial degree of the modes independently for all edges and independently for each of the fields \(w, \phi_1\) and \(\phi_2\). So \(w\) can be approximated by polynomials of one degree higher than the rotations, which is, in view of the Kirchhoff constrain for thin plates, favourable to overcome the locking problem.

To complete the description of all parts necessary for implementation of the algorithm in section 2, computable error indicators and estimators have to be defined.

Let \(A_i\) be the area of element \(i\), \(m_s\) the \(s\)-th diagonal element of
the material matrix $D_i$ in (10), $p_i$ the mean polynomial degree in element $i$, $l_{ik}$ the length of edge $r_k$ in element $i$ and $J_{iks}$ the jump of the $s$-th component of the stress resultant $\sigma$ in (8) across edge $k$ of element $i$. Then for each edge $r_k$ of element $i$ the following edge error indicator for the energy norm of the error can be computed \[ \lambda_{iks}^2 = \frac{A_i}{(24 p_i m_s l_{ik})} \int_{r_k} J_{iks}^2 \, dr \] \[ (17) \]

The element error indicator is then defined by
\[ \lambda_i^2 = \sum_{s=1}^{5} \sum_{k=1}^{4} \lambda_{iks}^2 \]

\[ (18) \]

For bilinear elements (i.e. $p_i = 1$) the error indicator (17), (18) was derived on the basis of the theory of Babuska 16. The correction factor $1/p_i$ for higher order elements was found on a heuristic basis as in 14.

Splitting the element error indicators into indicators for each stress component and each edge allows full control for each variable over the mesh. Depending on the stress components, the individual polynomial degree can be adjusted for each variable.

4. Shear force computation

It is a common characteristic of finite element computations based on Gaussian quadrature in the stiffness matrix calculation, that stresses are most reliably evaluated at integration points. However, practical applications, graphical result representation,(and also the a-posteriori error estimates (17) and (18)), need stress results on the element edges and in the nodes. Therefore most finite element programs postprocess stress resultants by extrapolation of integration point results to the nodes and by a local or global smoothing of nodal values.

On the other hand, it would be straightforward to compute derivatives of the displacements directly from the finite element solution anywhere in the element. Whereas this method
does often not work satisfactorily in the standard h-version of the FEM, it has been shown in \(^3\) that it yields very good results for the p- and hp-versions in the case of plane strain and stress problems. The same observation holds for the computation of bending moments in Reissner-Mindlin plate elements. Yet a direct evaluation fails in the case of the shear forces which are influenced not only by \(\theta_1, \theta_2\), but also by the first derivatives of \(w\).

The problems can be overcome by introducing a different shear force computation algorithm. Considering the differing orders of the Gauss integration in distorted elements, an interpolation algorithm on the basis of all integration points, strictly coupling the number of integration points to the degree of the interpolation, is not recommendable. Instead, we have chosen a least squares approximation to the integration point values for each element individually. The bending moments being of degree \(p(\theta) = 1 = p(w)^{-2}\) (where \(p(\theta)\) and \(p(w)\) are polynomial degrees for the approximation of the rotations and the displacement respectively), \(p(w)^{-2}\) seems to be reasonable for a shear force approximation. It is convenient to use the shape functions as approximation polynomials. Indeed, shear forces calculated thus prove to be very acceptable.

This shear force computation is obviously more costly than a direct evaluation from the derivatives of the shape functions. Yet this additional computational cost needs, (at least for thin plates ), only be spent for postprocessing finite element results after refinement and not for the computation of error indicators defined in (17). The shear force part to the total energy of the solution (and thus the contribution of shear force jumps to the error indicators) is small and can thus be neglected.

5. Numerical Examples

In the following numerical examples we will try to demonstrate that the hp-version of the FEM shows essentially the same features for Reissner-Mindlin plate problems as have been reported for potential and plain elasticity problems. These are:

- **robustness**

It will be shown that the p- and hp-version is robust against distortion of elements and against degeneration of Reissner-Mindlin theory for very thin plates, i.e. against the locking effect.

- **accuracy**

A rigorous accuracy evaluation of a numerical method can only be performed by comparing computed results with analytical
solutions. Yet, except for trivial cases, it is extremely hard to obtain analytic results in Reissner-Mindlin theory. So we will try to evaluate the accuracy by observing the ability of the hp-version to model accurately the behaviour of the solution near singular points.

- efficiency

Due to the large element matrices of the p-version it is clear that a computation with a comparable number of degrees of freedom is much more costly in the p- or hp-approach than with standard low degree elements. Yet it will be demonstrated that a comparable amount of computational resources yields dramatically improved results in the new approach. It is even more important in practical applications, that, due to the adaptivity of our approach, the engineering effort is also reduced drastically.

5.1 Distorted and thin plate elements (p-version)

In this section, we shall compare the high-order degree elements with standard elements presented in the literature\(^{18}\) which have been constructed with special emphasis on avoiding shear locking problems. To add some numerical difficulty, we used rather distorted elements, thus justifying their use in meshes created by the adaptive hp-refinement.

Shear locking will be tested by continuously increasing the thickness to length ratio of a plate and by comparing displacements to Kirchhoff's solution.

The plate presented here is a quadratic, clamped plate with uniform load. Fig. 3 shows the meshes used in the conventional analysis with 'heterosis' elements (8x8 and 6x6 elements, i.e. 483 and 255 dof, respectively) and the p-version mesh (2x2 elements). The polynomial degrees used in the p-type computation were \(p(w)=7\) (155 dof) and \(p(w)=8\) (219 dof). As explained in section 3, the polynomial degree \(p(\theta)\) was chosen to be \(p(\theta) = p(w) - 1\).

Figs. 4a and 4b show the maximum displacement, normed by the system parameters,

\[ w_{\text{norm}} = \frac{w_{\text{max}} \times \frac{E}{t} \times \frac{L}{l^4}}{p}, \]

(where \(w_{\text{max}}\) is the maximum displacement, \(E\) Young's modulus, \(p\) the load, and \(l\) the dimension of the plate) as a function of both the thickness ratio \(t/\) in logarithmic scale and the number of the degrees of freedom employed in the analysis. The Kirchhoff theory value of \(w\) is equal to 0.0152. Fig. 4a demonstrates the locking effect occurring with the conventional elements, whereas fig. 4b shows that the high polynomial-degree solution remains very stable even for a thickness ratio of 10**4. However, it should be mentioned that in the very thin
range beyond a ratio of $10^4$ the condition number of the stiffness matrix becomes large enough to cause convergence problems for the iterative equation solver used in our code. For these problems direct solvers should be preferable.

5.2 The clamped L-shaped plate (hp-version).

Our final example shows the results for the L-shaped plate clamped at the inner edges.

We will compare results of two fine uniform meshes with 75 and 300 (figure 5a) 'heterosis' elements with 735 resp. 2820 degrees of freedom to an hp-version result with only 273 degrees of freedom. The hp-mesh shown in figure 5b was created automatically by the refinement strategy defined in section 2 from a starting mesh consisting of only 3 elements. Figure 6 shows the bending moment along the cut (1-1) in figure 5a, figure 7 the shear force.

Obviously the hp-version results are by far better than the results of even the fine conventional mesh with more than 10 times the number of degrees of freedom. This can be seen from the behaviour of the stress resultants near the singularity but also from the results in a considerable distance from the reentrant corner. The pollution due to the singularity is big enough to introduce considerable error in the conventional solution. No such degeneration and no oscillations can be observed in the hp-version results. This is also interesting in view of our error indicators for the adaptive mesh refinement and polynomial degree control. As we used error indicators and estimators for the energy norm of the error, we could only expect optimal or near optimal results in this norm. But our examples show, that the pointwise quality of our results is superior, too.

A fair comparison between the two different methods has not to be drawn on the basis of the number of degrees of freedom but of the actual computational cost. The total CPU-time for the hp-computation was about 70% of the CPU-time for the standard computation on the fine mesh. Although a comparison of CPU-times is strongly implementation-dependent, these computations show that, at least if high quality results are desired, the hp-version is much more efficient than a standard finite element method.

Of course the conventional computation could be improved by a graded mesh, but the engineer's effort to create such a mesh is so big that, in practice, it is seldomly spent for such a 'simple' problem. Here, the adaptive hp-method is an excellent tool not only to reduce the mesh construction labour, but at the same time to facilitate quality control and to leave the engineer's time to critically analyse the computed data.
Conclusions

An hp-version of the finite element method for Reissner-Mindlin plate problems has been presented which shows highly accurate results and is robust against shear locking. A flexible data structure allows the implementation of an algorithm which adaptively increases the polynomial degree of the shape functions and refines the mesh towards singular points of the exact solution. Numerical results show that, as it was demonstrated earlier for potential and plane elastic problems, the method is superior to the standard h-version finite element method especially in cases where there are singularities in the exact solution. It offers the opportunity to release the engineer from the time consuming construction of a finite element mesh which not only describes the geometrical and physical properties of the problem but which is able to take the most of the profit out of the possibilities of the finite element method.

References


/9/ Rank, E., Babuska, I., 'An expert system for the optimal mesh design in the hp-version of the finite element method', Int. J. for Num. Meth. in Eng. 24, 2087-2106 (1987)


Fig. 1.: Meshes for an L-shaped domain
Fig. 2: Convergence in energy norm of the meshes shown in fig. 1.
Fig. 3.: Meshes used for the shear locking analysis:
a and b: meshes with conventional isoparametric
8-noded elements
c: mesh with high-polynomial-degree elements
Fig. 5.: Meshes used for the L-shaped plate
  a conventional mesh
  b mesh created by automatic refinement toward the singular point
Fig. 6: Bending moment creating stresses in y-direction at line 1-1 of the L-shaped plate;
1 and 2: conventional uniform meshes
3 graded hp-mesh
Fig. 7.: Shear forces Qy at line 1-1 of the L-shaped plate in fig. 5.
1 uni l. conventional uniform meshes
2 uni l. h-type mesh
3 p(w) = 5 (hp), 273 dof

1 735 dof, 75 elem.
2 2820 dof, 300 elem.
3 p(w) = 5 (hp), 273 dof