A $p$-version finite element approach for two- and three-dimensional problems of the $J_2$ flow theory with non-linear isotropic hardening

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SUMMARY

In this paper an implementation of a two- and three-dimensional $p$-version approach to the $J_2$ flow theory with non-linear isotropic hardening for small displacements and small strains is presented. Based on higher-order quadrilateral and hexahedral element formulations, a Newton–Raphson iteration scheme combined with a radial return algorithm is applied to find approximate solutions for the underlying physically non-linear model problem. Curved boundaries are taken care of with the blending function method, allowing an accurate representation of geometry with only a few $p$-elements. Numerical examples demonstrate, that the $p$-version supplies efficient and accurate approximations to this class of physically non-linear problems. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: $p$-version; finite element method; plasticity; $J_2$ flow theory; physically non-linear problems

1. INTRODUCTION

The finite element method can be classified into three groups: the $h$-version, the $p$-version and a combination of both, the $hp$-version. Whereas convergence in case of the $h$-version is obtained by local or global mesh refinement the $p$-version leaves the mesh unchanged and increases the polynomial degree of the shape functions locally or globally. For linear elliptic problems it was demonstrated by many authors [1–7], that the $p$-version of the finite element method leads to very efficient approximations, being often superior to the classical $h$-version approach. It was demonstrated by Szabó [8] that even in case of singularities in the exact solution, the $p$-version supplies in combination with a proper mesh design an exponential rate of convergence in energy norm in the preasymptotic range. An accuracy being acceptable

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for practical purposes can be readily obtained, if the mesh is refined towards points or lines of singularity or to resolve boundary layers. Another important feature of the $p$-version is its robustness with respect to locking effects. It was shown (e.g. References [6, 9]) for many problems of practical importance that the $p$-version is free of locking, if the polynomial degree is chosen to be higher than a certain order. To give an example, the shear locking effect in case of thin Reissner–Mindlin plates disappears if shape functions with a polynomial degree of $p \geq 4$ are chosen. Also the Poisson locking being an essential problem in elastoplastic computations is no longer observable, if the polynomial degree is chosen to be moderately high [7]. The asymptotic rate of convergence in energy norm for a $p$-extension is not affected when Poission's ratio $\nu$ tends to 1/2. In Reference [7] it is analytically and numerically shown however that the sum of normal stresses may have pointwise large errors if $\nu$ is very close to 1/2. However, the numerical examples presented in this paper demonstrate that a good pointwise convergence of stress components can be obtained even if the point of interest is located in the plastic region.

Up to now, most of the research on the $p$- and $hp$-version in structural mechanics considers linear elliptic problems only. For elastoplastic problems the only investigations of the $p$-version known to the authors are published in References [10–13]. In References [10, 12, 13] the deformation theory of plasticity (Hencky-type plasticity) was considered and in Reference [13] it was demonstrated by two-dimensional numerical examples that $p$-extensions are effective for controlling errors of discretization associated with elastic–plastic material behaviour. In Reference [12] the $p$-version was applied to approximate the solution of two-dimensional problems with sharp (but continuous) displacement gradients. For a two-dimensional benchmark problem, the $p$-version turned out to be superior even when being compared to an adaptive $h$-version finite element method [10]. In [11] the $p$-version was investigated for the physically more realistic model problem of ideal elastic–plastic $J_2$ flow theory. Again, it was demonstrated by two-dimensional numerical examples that the $p$-version yields clearly superior accuracy in comparison with its $h$-version counterparts.

In this paper we will show numerically that a higher order finite element method for two- as well as three-dimensional problems leads to an accurate and efficient discretization for more general physically non-linear models, including also cyclic loading. The chosen type of plasticity is the classical $J_2$ flow theory with small strains and displacements including non-linear isotropic hardening.

The paper is organized as follows: In Section 2 the model problem and the numerical treatment of the underlying set of differential-algebraic equations is presented. A short introduction into the $p$-version and a presentation of the chosen element formulations is given in Section 3. Numerical examples in two as well as in three dimensions demonstrate the efficiency of the $p$-version for the $J_2$ plasticity in Section 4. Finally, we conclude with a summary.

2. CLASSICAL $J_2$ FLOW THEORY WITH NON-LINEAR ISOTROPIC HARDENING

The physically non-linear model problem to be considered is the $J_2$ flow theory for small strains with non-linear isotropic hardening [14–17]. We assume that strains

$$\varepsilon = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

(1)
Table I. Classical $J_2$ flow theory with isotropic hardening.

1. Linear isotropic elastic stress–strain relationship:
   \[ \sigma = C : \varepsilon \]

2. Elastic domain in stress space:
   \[ \mathbb{E}_\varepsilon = \{ (\sigma, \varepsilon) \mid f(\sigma, \alpha) < 0 \} \]

3. Flow rule and hardening law:
   \[ \varepsilon^p = \gamma \frac{\text{dev}(\sigma)}{||\text{dev}(\sigma)||} \]
   \[ \dot{\varepsilon} = \gamma \sqrt{\frac{2}{3}} \]

4. Kuhn–Tucker loading/unloading conditions:
   \[ \gamma \geq 0, \quad f(\sigma, \alpha) < 0, \quad \gamma f(\sigma, \alpha) = 0 \]

5. Consistency condition:
   \[ \gamma \dot{f}(\sigma, \alpha) = 0 \]

are small and can be decomposed into an elastic and a plastic part

\[ \varepsilon = \varepsilon^e + \varepsilon^p \] (2)

Following [17] the differential-algebraic equations of the $J_2$ flow theory for small strains with isotropic hardening are summarized in Table I.

The stress tensor $\sigma$ is given by a linear isotropic elastic relationship, depending only on the elastic strains $\varepsilon^e$ and the tensor of elastic moduli

\[ C = \kappa \mathbf{I} \otimes \mathbf{I} + 2\mu [\mathbf{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I}] \] (3)

where $\kappa$ is the bulk modulus, $\mu$ is the shear modulus, $\mathbf{I} = \delta_{ik} \delta_{jl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ is the fourth-order unit tensor and $\mathbf{I} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ is the second-order unit tensor. In Table I the operator $\cdot : \cdot$ denotes the double contracted product of $\mathbf{C}$ with $(\varepsilon - \varepsilon^p)$, i.e. $\sigma_{ij} = C_{ijkl} (\varepsilon_{kl} - \varepsilon^p_{kl})$. Admissible stress states are defined by the von Mises yield criterion

\[ f(\sigma, \alpha) = ||\text{dev}(\sigma)|| - \sqrt{\frac{2}{3}} K(\alpha) \leq 0 \] (4)

where $|| \cdot || := \sqrt{\cdot : \cdot}$ is the Euclidean norm of a tensor, $\text{dev}[\cdot] := (\cdot) - \frac{1}{3} \text{tr}[\cdot] \mathbf{I}$ is the deviatoric part of a tensor and $\text{tr}[\cdot]$ denotes the trace operator. The elastic domain is defined as the interior of $\mathbb{E}_\varepsilon$ (where $f(\sigma, \alpha) < 0$) and the boundary of $\mathbb{E}_\varepsilon$ (where $f(\sigma, \alpha) = 0$) is referred to as the yield surface. States $f(\sigma, \alpha) > 0$ outside $\mathbb{E}_\varepsilon$ are non-admissible. The evolution of plastic strains is given by an associated flow rule. An internal variable $\alpha$ being often referred to as equivalent plastic strain describes the non-linear isotropic hardening

\[ K(\alpha) = \sigma_0 + h\alpha + (\sigma_\infty - \sigma_0)(1 - \exp(-\omega\alpha)) \] (5)

which is composed by a linear and an exponential function, where $\sigma_0$ is the initial yield stress, $h$ the linear hardening parameter, $\sigma_\infty$ the saturation stress and $\omega$ the hardening exponent. In Figure 1 $K(\alpha)$ is plotted for various sets of material parameters. $\gamma \geq 0$ is called consistency parameter and obeys the Kuhn–Tucker conditions, which are also known as loading/unloading conditions. The consistency condition states that either $\dot{f}(\sigma, \alpha) = 0$ or $\gamma = 0$. Plastic loading can only take place if $\gamma > 0$, therefore $\dot{f}(\sigma, \alpha) = 0$. For a detailed discussion on the consistency and Kuhn–Tucker conditions see Reference [17]. Summarizing, the described material model contains six parameters which are listed in Table II.
Figure 1. Linear and non-linear isotropic hardening.

<table>
<thead>
<tr>
<th>Number</th>
<th>Material parameter</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Bulk modulus</td>
<td>$\kappa$ (MPa)</td>
</tr>
<tr>
<td>2</td>
<td>Shear modulus</td>
<td>$\mu$ (MPa)</td>
</tr>
<tr>
<td>3</td>
<td>Initial yield stress</td>
<td>$\sigma_0$ (MPa)</td>
</tr>
<tr>
<td>4</td>
<td>Saturation stress</td>
<td>$\sigma_\infty$ (MPa)</td>
</tr>
<tr>
<td>5</td>
<td>Linear hardening</td>
<td>$h$ (MPa)</td>
</tr>
<tr>
<td>6</td>
<td>Hardening exponent</td>
<td>$\omega$ dimensionless</td>
</tr>
</tbody>
</table>

Table II. Material parameters for a non-linear model problem.

Of course the stresses do not only have to satisfy the constitutive equations listed above but also the equilibrium conditions. The weak formulation of the equilibrium conditions of a three-dimensional solid body with domain $\Omega \subset \mathbb{R}^3$ and boundary $\Gamma = \Gamma_D \cup \Gamma_N$ ($\Gamma_D$ being the Dirichlet, $\Gamma_N$ the Neumann boundary) reads:

Find the displacement field $u \in V = \{v(x) \in [H^1(\Omega)]^3: v = 0 \text{ on } \Gamma_D\}$ satisfying the (homogeneous) Dirichlet boundary conditions, such that

$$
\int_\Omega \varepsilon(v) : \sigma(u) \, d\Omega = \int_\Omega v \cdot f \, d\Omega + \int_{\Gamma_N} v \cdot \tilde{t} \, d\Gamma \quad \forall v \in V
$$

where $f$ are body forces and $\tilde{t} = \sigma n$ are surface tractions on $\Gamma_N$ with $n$ being the unit normal vector of the surface element $d\Gamma$. $H^1(\Omega)$ denotes the Sobolev space of functions possessing square integrable derivatives. Due to the non-linear stress–strain relationship depending on the strain history the weak formulation is a non-linear functional which has to be solved incrementally. We therefore compute a sequence of equilibrated load steps by applying for each step $[t(n),t(n+1)]$ the Newton–Raphson method in order to linearize the weak formulation:

$$
\int_\Omega \varepsilon(v) : C^{(i)}_{(n+1)} : \varepsilon(\Delta u^{(n+1)}) \, d\Omega = \int_\Omega v \cdot f_{(n+1)} \, d\Omega + \int_{\Gamma_N} v \cdot \tilde{t}_{(n+1)} \, d\Gamma - \int_\Omega \varepsilon(v) : \sigma_{(n+1)}^{(i)}(u) \, d\Omega
$$

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\[ u^{(i+1)}_{(n+1)} = u^{(i)}_{(n+1)} + \Delta u^{(i+1)} \]

\[ C^{(i)}_{(n+1)} := \frac{\partial \sigma^{(i)}_{(n+1)}}{\partial \varepsilon^{(i)}_{(n+1)}} \]

\((\cdot)^{(i)}_{(n+1)}\) denotes a variable at the \(i\)th Newton–Raphson iteration during the load step in \([t_{(n)}, t_{(n+1)}]\). In the context of rate-independent plasticity, \(t_{(n)}\) denoted pseudo-time, ordering the steps being applied during the incremental loading. A new equilibrated configuration is found, if a proper convergence criterion is satisfied (e.g. \(|\Delta u^{(i+1)}/u^{(i+1)}_{(n+1)}| < \delta_u\)). During the Newton–Raphson iteration the actual stress state \(\sigma^{(i)}_{(n+1)}\) as well as the algorithmic tangent modulus \(C^{(i)}_{(n+1)}\) have to be recomputed in each step \(i\).

The computation of the stresses is carried out using an algorithm being based on an elastic-predictor/plastic-corrector scheme [17–19]. Therefore, in a first step the elastic predictor is computed

\[ \text{dev}[\sigma^{\text{trial}}_{(n+1)}] = 2\mu \text{dev}[\varepsilon^{(i)}_{(n+1)} - \varepsilon^p_{(n)}] \]

If

\[ f(\sigma^{\text{trial}}_{(n+1)}, \alpha_{(n)}) = ||\text{dev}[\sigma^{\text{trial}}_{(n+1)}]|| - \sqrt{\frac{2}{3}} (\sigma_0 + h\alpha_{(n)} + (\sigma_\infty - \sigma_0)(1 - \exp(-\omega\alpha_{(n)}))) \leq 0 \]

the stresses are admissible and we set

\[ \sigma^{(i)}_{(n+1)} = \kappa \text{tr}[\varepsilon^{(i)}_{(n+1)}] I + \text{dev}[\sigma^{\text{trial}}_{(n+1)}] \]

The variables \(\varepsilon^p_{(n+1)} = \varepsilon^p_{(n)}\) and \(\alpha^{(i)}_{(n+1)} = \alpha_{(n)}\) remain constant.

If \(f(\sigma^{\text{trial}}_{(n+1)}, \alpha_{(n)}) > 0\) the actual stress state is non-admissible and therefore the plastic corrector has to be applied. By integrating the local constitutive equations (Table 1) with given initial conditions based on the converged values \((\cdot)_{(n)}\) at the end of the last time step a new state \(\{\sigma^{(i)}_{(n+1)}, \varepsilon^{p_{(n+1)}}, \alpha^{(i)}_{(n+1)}\}\) is determined. The integration is carried out by applying an implicit backward-Euler difference scheme leading to the following non-linear system:

\[ \sigma^{(i)}_{(n+1)} = \kappa \text{tr} [\varepsilon^{(i)}_{(n+1)}] I + 2\mu \text{dev} \left[ \varepsilon^{(i)}_{(n+1)} - \varepsilon^{p_{(n+1)}} \right] \]

\[ \varepsilon^{p_{(n+1)}} = \varepsilon^p_{(n)} + \Delta \gamma^{(i)} \frac{\text{dev}[\sigma^{(i)}_{(n+1)}]}{||\text{dev}[\sigma^{(i)}_{(n+1)}]||} \]

\[ \alpha^{(i)}_{(n+1)} = \alpha_{(n)} + \sqrt{\frac{2}{3}} \Delta \gamma^{(i)} \]

\[ f(\sigma^{(i)}_{(n+1)}, \alpha^{(i)}_{(n+1)}) = 0 \]

where Equation (14) is a constraint, ensuring that the new stress state fulfils the yield condition. An efficient way of solving Equations (11)–(14) is to reduce the system to one non-linear
equation in $\Delta \gamma$

\[
f(\Delta \gamma^{(i)}) = 0 = \|\text{dev}[\sigma_{e_{(e+1)}}]\| - 2\mu \Delta \gamma^{(i)} - \sqrt{\frac{2}{3}} \left( \sigma_0 + h \left( \alpha(n) + \sqrt{\frac{2}{3}} \Delta \gamma^{(i)} \right) \right)
+ (\sigma_0 - \sigma_0) \left( 1 - \exp \left( -\omega \left( \alpha(n) + \sqrt{\frac{2}{3}} \Delta \gamma^{(i)} \right) \right) \right) \right)
\]  

(15)

A general strategy to reduce the non-linear algebraic system resulting from the numerical integration of the constitutive equations for isotropic as well as kinematic hardening to a scalar equation can be found e.g. in References [17, 18]. Equation (16) can be easily solved by applying a local Newton iteration.

Considering a linear isotropic hardening law, e.g. $K(\alpha) = \sigma_0 + h\alpha$ the solution of $f(\Delta \gamma^{(i)}) = 0$ leads to a closed-form

\[
\Delta \gamma^{(i)} = \frac{f(\sigma_{e_{(e+1)}}^\text{trian}, \alpha(n))}{2\mu + \frac{2}{3}h}
\]  

(16)

For elastic perfectly plastic behaviour, i.e. $h = 0$, Equation (16) reduces to the classical radial return method [19]. In Table III the outlined algorithm for $J_2$ flow theory with non-linear isotropic hardening for both the three-dimensional and plane strain case is summarized.

The spatial discretization of Equation (7) is performed with the $p$-version of the finite element method. The element formulations, applied in this paper will be briefly summarized in the following section.

3. THE $p$-VERSION OF THE FINITE ELEMENT METHOD

Our $p$-version implementation is based on the one-dimensional hierarchic basis, proposed by Szabó and Babuška [7]. Using this basis, Ansatz functions can be implemented up to any desired polynomial degree. Two- and three-dimensional Ansatz functions are constructed by simply building the tensor product of one-dimensional hierarchic shape functions.

The two-dimensional shape functions (for quadrilaterals) can be classified into three groups: nodal modes, edge modes and internal modes. Two different types of Ansatz spaces have been implemented: the trunk space $S^{P_3}_{\text{is}}\cdot P_3^e(\Omega^h_3)$ and the tensor product space $S^{P_3}_{\text{ps}}\cdot P_3^e(\Omega^h_3)$. The two-dimensional examples, presented in this paper are based on the trunk space $S^{P_3}_{\text{is}}(\Omega^h_3)$.

For a three-dimensional discretization we consider a hexahedral element formulation also based on the one-dimensional Ansatz functions. Three different types of Ansatz spaces have been implemented: the trunk space $S^{P_3}_{\text{is}}\cdot P_3^e\cdot P_3^e(\Omega^h_3)$, the tensor product space $S^{P_3}_{\text{ps}}\cdot P_3^e\cdot P_3^e(\Omega^h_3)$ and the space $S^{P_3\cdot P_3\cdot P_3}(\Omega^h_3)$. A detailed description of the three Ansatz spaces is given in Reference [20] and the literature listed there. For the definition of the spaces $S^{P_3}_{\text{is}}\cdot P_3^e\cdot P_3^e(\Omega^h_3)$ and $S^{P_3\cdot P_3\cdot P_3}(\Omega^h_3)$ see also Szabó and Babuška [7]. In three dimensions, the shape functions can be classified into four groups: nodal modes, edge modes, face modes and internal modes (see [7, 20]). In this paper we will restrict our investigations to the trunk space $S^{P_3}_{\text{is}}\cdot P_3^e\cdot P_3^e(\Omega^h_3)$.
1. Compute trial elastic stress:
   \[ \text{dev}[\sigma_{(n+1)}^{\text{trial}}] = 2\mu \text{dev}[^{\text{strain}}\varepsilon_{(n)}] - \varepsilon_{(n)}^p \]

2. Check yield condition:
   \[ f(\sigma_{(n+1)}^{\text{trial}}, \varepsilon_{(n)}) = \| \text{dev}[\sigma_{(n+1)}^{\text{trial}}] \| - \sqrt{\frac{2}{3}} (\sigma_0 + h\varepsilon_{(n)} + (\sigma_\infty - \sigma_0)(1 - \exp(-\omega\varepsilon_{(n)}))) \]
   \[ \text{IF } f(\sigma_{(n+1)}^{\text{trial}}, \varepsilon_{(n)}) \leq 0 \]
   \[ \varepsilon_{(n+1)}^{(i)} = \varepsilon_{(n)}^p \]
   \[ \varepsilon_{(n+1)}^{(i)} = \varepsilon_{(n)}^p \]
   \[ \sigma_{(n+1)}^{(i)} = \kappa \text{tr}[^{(i)}\varepsilon_{(n+1)}]I + \text{dev}[\sigma_{(n+1)}^{\text{trial}}] \]
   \[ C_{(n+1)}^{(i)} = \kappa I \otimes I + 2\mu [I - \frac{1}{3}I \otimes I] \]
   \[ \text{EXIT} \]

3. Compute consistency parameter \( \Delta \gamma^{(i)} \):
   \[ f(\Delta \gamma^{(i)}) = 0 = \| \text{dev}[\sigma_{(n+1)}^{\text{trial}}] \| - 2\mu \Delta \gamma^{(i)} - \sqrt{\frac{2}{3}} \left( \sigma_0 + h \left( \varepsilon_{(n)} + \sqrt{\frac{2}{3}} \Delta \gamma^{(i)} \right) \right. \]
   \[ \left. + (\sigma_\infty - \sigma_0) \left( 1 - \exp \left( -\omega \left( \varepsilon_{(n)} + \sqrt{\frac{2}{3}} \Delta \gamma^{(i)} \right) \right) \right) \right) \]

4. Update internal variable, plastic strain and stress:
   \[ n_{(n+1)}^{(i)} = \frac{\text{dev}[\sigma_{(n+1)}^{\text{trial}}]}{\| \text{dev}[\sigma_{(n+1)}^{\text{trial}}] \|} \]
   \[ \varepsilon_{(n+1)}^{(i)} = \varepsilon_{(n)} + \sqrt{\frac{2}{3}} \Delta \gamma^{(i)} \]
   \[ \varepsilon_{(n+1)}^{(i)} = \varepsilon_{(n)} + \Delta \gamma^{(i)} n_{(n+1)}^{(i)} \]
   \[ \sigma_{(n+1)}^{(i)} = \kappa \text{tr}[^{(i)}\varepsilon_{(n+1)}]I + \text{dev}[\sigma_{(n+1)}^{\text{trial}}] - 2\mu \Delta \gamma^{(i)} n_{(n+1)}^{(i)} \]

5. Compute consistent elastoplastic tangent moduli:
   \[ C_{(n+1)}^{(i)} = \kappa I \otimes I + 2\mu \theta_{(n+1)} [I - \frac{1}{3}I \otimes I] - 2\mu \bar{\theta}_{(n+1)} n_{(n+1)}^{(i)} \otimes n_{(n+1)}^{(i)} \]
   \[ \theta_{(n+1)} = 1 - \frac{2\mu \Delta \gamma^{(i)}}{\| \text{dev}[\sigma_{(n+1)}^{\text{trial}}] \|} \]
   \[ \bar{\theta}_{(n+1)} = \left( 1 + \frac{h + \omega (\sigma_\infty - \sigma_0) \exp(-\omega \varepsilon_{(n+1)}^{(i)})}{3\mu} \right)^{-1} \left( 1 - \theta_{(n+1)} \right) \]

Due to the fact, that internal degrees of freedom are purely local to the element they are eliminated by static condensation. It was shown by different authors [21–23], that this condensation of internal degrees of freedom can be interpreted as an efficient preconditioning. To solve the remaining part of the overall equation system a PCG-solver with SSOR preconditioning is applied.

Figure 2. Perforated square plate under plane strain condition with cyclic loading.

<table>
<thead>
<tr>
<th>Number</th>
<th>Material parameter</th>
<th>Set 1</th>
<th>Set 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Bulk modulus ( \kappa )</td>
<td>164206.0 (MPa)</td>
<td>164206.0 (MPa)</td>
</tr>
<tr>
<td>2</td>
<td>Shear modulus ( \mu )</td>
<td>80193.8 (MPa)</td>
<td>80193.8 (MPa)</td>
</tr>
<tr>
<td>3</td>
<td>Initial yield stress ( \sigma_0 )</td>
<td>450.0 (MPa)</td>
<td>450.0 (MPa)</td>
</tr>
<tr>
<td>4</td>
<td>Saturation stress ( \sigma_\infty )</td>
<td>0.0 (MPa)</td>
<td>715.0 (MPa)</td>
</tr>
<tr>
<td>5</td>
<td>Linear hardening ( h )</td>
<td>0.0 (MPa)</td>
<td>129.24 (MPa)</td>
</tr>
<tr>
<td>6</td>
<td>Hardening exponent ( \omega )</td>
<td>0.0 (dimensionless)</td>
<td>16.93 (dimensionless)</td>
</tr>
</tbody>
</table>

4. NUMERICAL EXAMPLES

4.1. Perforated square plate under plane strain condition with cyclic loading

The first numerical example to be considered is a perforated square plate under plane strain condition with cyclic loading (see Figure 2). This problem was defined by Stein as a benchmark of the German research project 'Adaptive finite element methods in applied mechanics' (see Reference [24]).

A quarter of a square plate with a central circular hole is loaded by a traction \( p = 100 \) (MPa) which is scaled with a factor \(-4.5 \leq \lambda \leq 4.5\) in 205 load steps as plotted in Figure 2. At the lower and right side of the plate symmetry conditions are imposed.

Two different sets of material parameters are to be considered (see Table IV). The first set corresponds to an elastic perfectly plastic behaviour whereas the second one includes also non-linear isotropic hardening. The plate is meshed with 4 (mesh A) and 48 (mesh B) quadrilaterals as depicted in Figure 3. Considering a \( p \)-extension for a linear elastic problem, mesh A would lead to an efficient discretization while mesh B would be by far too fine. Taking advantage of the blending function method (see [7, 20, 25]) the geometry of the hole is exactly represented.
Figure 3. Perforated square plate meshed with four and 48 quadrilaterals.

Figure 4. Mesh A: displacement $u_x$ at point 2 (upper part: without hardening, lower part: with hardening).

Figure 5. Mesh B: displacement $u_x$ at point 2 (upper part: without hardening, lower part: with hardening).

All $p$-version computations are based on the trunk space $S_3^{p_c,p_q}(\Omega^h)$ with $p = p_c = p_q = 5, 10, 15, 20$ for mesh A and $p = p_c = p_q = 4, 6, 8, 10$ for mesh B. As a reference solution we use the results having been obtained by Wieners [26] with 65536 Q2/P1 elements and a corresponding number of 197633 unknowns. Of interest are the following physical quantities: the displacement component $u_x$ at point 2 (Figures 4, 5), the stress component $\sigma_{yy}$ at
Figure 6. Mesh A: stress $\sigma_{yy}$ at point 2 (upper part: without hardening, lower part: with hardening).

Figure 7. Mesh B: stress $\sigma_{yy}$ at point 2 (upper part: without hardening, lower part: with hardening).

point 2 (Figures 6, 7) and the principal stress $\sigma_{II}$ at point 7 (Figures 8, 9). The results for the elastic perfectly plastic behaviour are pictured in the upper parts whereas the results in case of non-linear isotropic hardening are displayed in the lower parts of Figures 4–9.

From Figures 4–9 it is evident that in case of isotropic hardening the approximations converge faster to the reference solution than in case of the elastic perfectly plastic model. This behaviour is related to the fact, that the stress–strain relationship in case of hardening is smoother compared to the elastic perfectly plastic one. Furthermore it is obvious, that an accurate approximation based on the coarse mesh A requires a very high polynomial degree of $p = 20$. Considering the results of the finer mesh B, it is evident that the $p$-version supplies—even in the case of elastic perfectly plastic behaviour—a very accurate approximation, when a polynomial degree of $p \geq 8$ is chosen.

To integrate the element stiffness matrices an integration scheme with $n_{Gp} = (p + 1) \times (p + 1)$ Gaussian points was applied. Consequently, as $p$ is raised the number of integration points grows in a quadratic manner and therefore the geometric resolution of the plastic range is raised. The influence of the integration order with respect to the accuracy of high order elements is investigated in Reference [10] for the deformation theory of plasticity. In Figures 10 and 11 the integration points where yielding occurs are plotted for the last load step with...
Figure 8. Mesh A: Principal stress $\sigma_{II}$ at point 7 (upper part: without hardening, lower part: with hardening).

Figure 9. Mesh B: Principal stress $\sigma_{II}$ at point 7 (upper part: without hardening, lower part: with hardening).

Figure 10. Integration points where yielding occurs (left part: without hardening, right part: with hardening).
\( \lambda = 0.0 \) for meshes A and B, respectively. As expected, the plastic zone of the plate with non-linear isotropic hardening is smaller than the one with elastic perfectly plastic behaviour. Furthermore it is obvious, that the plastic zone predicted by mesh A with \( p = 20 \) is very similar to the one predicted by mesh B with \( p = 10 \).

In Table V the computational cost for the perforated square plate under plane strain condition with cyclic loading (205 load steps) and non-linear isotropic hardening are listed. All simulations were performed on a COMPAQ XP1000 machine (alpha processor ev6 21264 with 500 Mhz). It is evident that a moderately fine discretization with 48 quadrilaterals leads to a more efficient discretization when being compared to the mesh with four elements. This is due to the fact that the amount of work, related to the computation of an element stiffness matrix grows fast with the corresponding polynomial degree \( p \) when a standard Gaussian quadrature scheme is used. Considering the \( p \)-version in two dimensions, the number of degrees of freedom \( n_{\text{dof}} \) and the number of Gaussian points \( n_{\text{Gp}} \) are both proportional to \( p^2 \). To form the matrix product \( B^T CB \) at one Gaussian point \( n_k \approx O(p^4) \) floating point operations are needed. Ignoring the numerical effort of evaluating the shape functions—being small compared to the one related to multiply \( B^T CB \) at each Gaussian point—the total number of floating point operations to integrate a stiffness matrix equals \( n_{\text{Gp}} \cdot n_k \) and is therefore proportional to \( p^6 \). Several strategies have been suggested to reduce the numerical effort related to the integration of \( p \)-version finite element stiffness matrices [27, 28]. A reduction of computational time

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**Table V. Computational cost for a computation with 205 load steps including isotropic hardening.**

<table>
<thead>
<tr>
<th>Degree</th>
<th>Mesh A</th>
<th>CPU time</th>
<th>Mesh B</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Degrees of freedom</td>
<td></td>
<td>Degrees of freedom</td>
<td></td>
</tr>
<tr>
<td>( p )</td>
<td>116</td>
<td>0 m 20 s</td>
<td>4</td>
<td>864</td>
</tr>
<tr>
<td>10</td>
<td>416</td>
<td>5 m 56 s</td>
<td>6</td>
<td>1776</td>
</tr>
<tr>
<td>15</td>
<td>916</td>
<td>45 m 16 s</td>
<td>8</td>
<td>3072</td>
</tr>
<tr>
<td>20</td>
<td>1616</td>
<td>3 h 32 m 47 s</td>
<td>10</td>
<td>4752</td>
</tr>
</tbody>
</table>

could therefore be expected after implementation of one of these methods. Having in mind, that for every load step a Newton–Raphson scheme with up to five iterations is performed, the total time for solving the problem based on mesh B—even with \( p = 10 \)—is still rather low.

4.2. Thick-walled plate with circular hole

As a second non-linear problem consider a thick-walled plate with circular hole under monotonous load (see Figure 12). This problem was defined by Stein as a benchmark of the German research project ‘adaptive finite element methods in applied mechanics’ (see Reference [24]).

The material parameters corresponding to set 2 are listed in Table IV. Due to symmetry only an eighth of the system has to be discretized. A mesh consisting of 48 hexahedral elements in conjunction with the trunk space \( \mathcal{S}_{\text{tr}}^{{\mathcal{P}_h}^2} \) (Figure 13) was used to perform a series of computations with \( p = p_e = p_h = p_c = 1, \ldots, 8 \). The curved boundary of the hole was taken care of with the blending function method (see References [7, 20, 25]). The load was raised monotonously within 61 load steps up to a factor \( \lambda = 4.15 \).

The first result to be considered is the displacement component \( u_r \) at point 2 for \( 1.0 \leq \lambda \leq 4.15 \) (see Figure 14). As a reference solution we use the results of Wieners [26] having
Plate 1. Von Mises stress and Gaussian points where yielding occurs.
been obtained with a fine mesh consisting of 1048576 Q1P0 hexahedral elements resulting in an equation system of 3,368,499 unknowns. Comparing the results of the \( p \)-version with the reference solution, only very small deviations are visible.

In Figure 15 the stress component \( \sigma_{yy} \) at point 2 based on an approximation with \( p = 8 \) and a corresponding number of 15368 degrees of freedom for all load steps is plotted. The source of oscillations of \( \sigma_{yy} \) at \( \lambda \approx 2.3 \) is related to a plastic zone crossing the interface of two elements.

To get an impression of the error related to \( \sigma_{yy} \) consider Figure 16. The maximum relative error is 1.1 per cent being very accurate for a three-dimensional non-linear computation.

The significance of a three-dimensional computation is proven in Plate 1 where the \textit{von Mises} stress and the Gaussian points where yielding occurs are plotted for the last load step \( \lambda = 4.15 \), showing the variation of the plastic region over the plate thickness.

5. CONCLUSION

In this paper a \( p \)-version approach for the \( J_2 \) flow theory with non-linear isotropic hardening in two as well three dimensions is presented. The underlying differential-algebraic equations of the \( J_2 \) plasticity are treated with a radial return algorithm. The constitutive equations are integrated with a backward-Euler difference scheme and the resulting non-linear system is reduced to one non-linear equation from which the consistency parameter is determined. The local radial return algorithm for the pointwise integration of the constitutive equations is embedded in a global Newton–Raphson iteration. Due to the robustness of the \( p \)-version against Poisson locking, a pure displacement formulation is chosen. Based on higher-order quadrilateral and hexahedral element formulations including the blending function method in order to exactly represent curved boundaries, it is numerically demonstrated that the \( p \)-version is an accurate and efficient discretization strategy for the investigated physically non-linear model problem.

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REFERENCES


