Adaptive Boundary Element Methods

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ADAPTIVE BOUNDARY ELEMENT METHODS

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SUMMARY

Boundary element methods are now a well understood and widely used tool in computational mechanics. Although the mesh construction is, due to the dimensional reduction, considerably easier than in finite element computations, there is still a need for reliable rules and tools to construct meshes and assign polynomial degrees in a reasonable and optimal way. This is, where adaptive methods start. In so-called feedback procedures, the approximating space is 'extended' in order to optimize the accuracy of the boundary element solution for a certain number of degrees of freedom. Adaptive methods are already well established in the finite element field, with quite well developed mathematical theory and very promising numerical results. In boundary element methods, on the other hand, there is nearly no theory available and only few numerical results on adaptive procedures have been reported. So this paper will review the achievements in adaptive finite element methods, will try to 'transform' some of the basic results to the boundary element situation and will show the possibilities and advantages of an adaptive BIE for some numerical examples.

1. INTRODUCTION

Since Babuska's basic work on adaptive finite element methods¹ there has been extensive research in this promising new field of 'intelligent' software in computational mechanics. For a survey see for example².

The basic ideas of adaptive finite element or boundary element methods are the following. The computation starts on a coarse mesh with low polynomial order of the shape functions, thus the input data of the user is kept to a minimum. From this first computation a posteriori information is extracted, i.e. some information about the accuracy of this first approximation
is computed. Usually some estimation of the magnitude and distribution of the error in a specified norm is of interest. With these estimations an extension process can be started. This means that the finite element space is enlarged. If the mesh is refined locally, this extension process is called h-version, an increase of the polynomial order of the shape functions is called p-version and a combination of both, i.e. a simultaneous refinement and increase of the order is called hp-version. There is already rather extensive mathematical theory on adaptive finite element methods \(^3,4,5,6\). It has been shown, that an optimal refinement of the finite element mesh, i.e. an adaptive h-version removes the influence of singularities in the exact solution on the rate of convergence of the approximation. The p-version, on the other hand, converges exponentially, if the exact solution is smooth, and converges always faster than a uniform refinement with fixed polynomial degree. Finally, a proper combination of mesh refinement and increase of polynomial order in an hp-version converges exponentially even in the presence of singularities in the exact solution. This has been shown theoretically in \(^7\) and in a finite element expert system \(^8,9\) the practical implementation has been demonstrated. Recently, a fully adaptive hp-version for linear potential, elasticity and plate problems has been implemented \(^10\), yielding superior accuracy with minimal input effort for the user.

Research on boundary element methods often has tried to profit from achievements obtained in finite element analysis, so it is quite natural to try to get similar improvement using adaptive BEM which were obtained in adaptive FEM. Yet, the mathematical foundation for adaptive boundary element methods, i.e. reliable \textit{a posteriori} estimations, is nearly completely missing. Only for a Galerkins method for the direct BEM in potential problems using constant elements some mathematical results have been obtained \(^17\). So most of the work on adaptive BEM is based on heuristic arguments, nevertheless yielding rather promising results.

The p-version of the BEM has been studied by Alarcon et al \(^11-13\) for potential problems in two and three dimensions. He uses refinement indicators similar to those derived by Peano \(^14\) for finite element applications. The principle idea is that for each possible new mode to be introduced in the finite element space it is estimated, how large the decrease of the error in energy might be. In \(^15\) a so-called p-method for the BEM is studied, which means that the number of degrees of freedom is kept fix but the nodes of the boundary discretization are repositioned so that the error is reduced. In \(^16\) Katz reports on an adaptive h-method based on a symmetric Galerkin formulation.
He uses error indicators similar to those developed in 17 and shows the practical advantages of an adaptive refinement. Other results on adaptive boundary elements have been reported in 18.

Recently Wendland has proposed a combined hp-version for the BEM 19, making the conjecture that a proper combination of refinement and increase of the polynomial degree would yield exponential convergence even to nonsmooth exact solutions, i.e. the same behaviour which is observed in FEM.

In the next chapter some basic terms of adaptive methods are reviewed and optimal strategies of an hp-version in the finite element method are discussed. Chapter 3 defines the model problem for the BEM application and gives a simple a posteriori estimation. Chapter 4 discusses some considerations for the implementation of an adaptive p- and hp-version BEM. Finally chapter 5 shows in two numerical examples that the suggested hp-implementation provides the desired exponential rate of convergence and yields results of superior accuracy with minimal human and computational effort.

2. ADAPTIVE METHODS

Let

\[ Lu = f \]  \hspace{1cm} (1)

be an operator equation with a linear differential or integral operator \( L \) and the unknown function \( u \) of some space \( H \) which shall be approximated by \( U \in S_1 \subset H \)

\[ L U = f \]  \hspace{1cm} (2)

We will assume that \( U \) has the typical form of finite element or boundary element approximations, i.e.

\[ U = \sum_{i=1}^{n} U_i N_i(x) \]  \hspace{1cm} (3)

with polynomial shape functions \( N_i \), \( i \) ranging over the 'elements' of the approximation. The error \( e = U - u \) shall be measured in some norm \( \| e \| \). \( \eta \) is called an error estimator for \( \| e \| \), if there are two constants \( C_1 \) and \( C_2 \), which are independent of the particular solution \( u \) and the finite element resp. the boundary element mesh so that the inequality

\[ C_1 \eta \leq \| e \| \leq C_2 \eta \]  \hspace{1cm} (4)
holds.

If the constants \( C_1 \) and \( C_2 \) are close to 1, the error estimator \( \eta \) gives a measure of the magnitude of the approximation error \( u - u_h \). Property (4) can usually be proven only under rather restrictive assumptions to the operator \( L \) and the approximation spaces \( S_j \). Yet in finite element methods error estimators have been developed for which the satisfaction of (4) is observed numerically for a wide class of problems.

**Error indicators** \( \lambda_i \) are related to error estimators by

\[
\eta^2 = \sum_{i=1}^{n} \lambda_i^2
\]

(5)

thus indicating the influence of a particular element \( i \) to the overall error of the approximation.

To be of any practical value, \( \lambda_i \) and \( \eta \) have to be computed only out of general properties of the operator \( L \) and the special approximation \( U \).

Once \( \lambda_i \) are known, the approximation space \( S_j \) can be extended to a space \( S_{j+1} \) in such a way that the error reduction is maximal. If this extension process is performed by refining the mesh, i.e. changing the mesh size parameter \( h \), one speaks of an \( h \)-version, an increase of the polynomial degree \( p \) of the shape functions \( N_i \) is the \( p \)-version, a combination of both the \( hp \)-version.

As this extension is controlled by a preliminary computation on the space \( S_j \), this process is called feedback procedure. This feedback can be performed several times, i.e. after the computation on \( S_{j+1} \) another \( a \) posteriori estimation is performed yielding the information for an extension to \( S_{j+2} \) and so on, until the estimated accuracy is below some prespecified value. If the feedback is done in an optimal way, i.e. yields best possible convergence properties, the process is called an adaptive method.

Let us now review some of the basic results on optimal strategies for an \( hp \)-extension in FEM. To be specific, let \( L \) in (1) be a linear elliptic differential operator on \( \Omega \subset \mathbb{R}^2 \). Then the keystone for the construction of an \( hp \)-version with the optimal exponential rate of convergence is a detailed understanding of the properties of the \( p \)-version in different situations. The first case to be considered is where the exact
solution \( u \) is smooth. Then a pure p-version converges exponentially to the exact solution:

\[
\|e\|_E \lesssim Ck^{-\alpha}N(p)^{1/2} \quad p \to \infty
\]

(6)

with the energy norm \( \|e\|_E \) of the error, positive constants \( C \), \( k \) and \( \alpha \) and the number of degrees of freedom \( N(p) \) of the approximation for polynomial degree \( p \).

The second case is where the exact solution has a singularity of the type

\[
u = u_0 + \sum_{i=1}^{\infty} C_i r^{\lambda_i} g_i(\theta)
\]

(8)

\( u_0 \) is a smooth function, \((r,\theta)\) are polar coordinates centered at the singularity, \( \lambda_i \) shall be ordered increasingly, \( C_i \) are constants, the 'stress intensity factors', and \( g_i \) smooth functions. In this case the pure p-version converges algebraically with a rate of convergence governed by \( \lambda_1 \).

For a fixed mesh the convergence behaviour is shown in curve (a) of figure 1. An inverted S-curve is observed which means that in the preasymptotic range (curved down) the p-version converges exponentially, then levelling off to the algebraic convergence (straight line). To achieve optimal accuracy for a certain number of degrees of freedom it is desirable to stay in the preasymptotic range of the convergence. This can be done by 'switching' at the right moment to a different mesh. Consider the sequence of geometric refinements as shown in figure 2(a-c) with 0,1 and 2 refinement layers towards the singular reentrant corner of the L-shaped domain. The convergence of the p-version on meshes 2(b) and (c) is shown in curve (b) resp. (c) of figure 1. Again, each of these curves is an inverted S, yet shifted compared to the curve for mesh 2(a). An hp-version switches now from mesh 2(a) to 2(b) and 2(c) at the intersection of curves (a) and (b) resp. (b) and (c), staying on the lower left envelope of all curves. The envelope itself is 'curved down', i.e. yields the desired exponential rate of convergence. For details and the proof of these properties see 7.

The choice of the right combination of polynomial degree and geometric mesh refinement at various singularities has been treated in a finite element expert system 8,9. An adaptive approach, which will also be the starting point of the adaptive hp-version for the BEM to be presented here, has been discussed
The algorithm consists of the following steps:

Step 1: Choose a basic mesh, which is just fine enough to describe geometry, boundary conditions and loads of the problem.
Step 2: Separate the elements of the basic mesh into two parts, those, where the exact solution is expected to be smooth (called non-critical elements) and those adjacent to a singular point of the exact solution, e.g. reentrant corners, points of change of boundary conditions etc. (called critical elements).
Step 3: Assign polynomial degree \( p=1 \) to each element.
Step 4: Perform a FEM-computation and compute error indicators for each element.
If the accuracy estimated by the error estimator is sufficient, STOP.
Step 5: For each element decide if the error indicator is above a prespecified level, i.e. if the accuracy has to be improved. If yes, then
for noncritical elements increase the polynomial degree by 1,
for critical elements refine geometrically towards the singularity in this element.

This algorithm can be transformed immediately to the boundary element method. The basic mesh describes again only geometry and boundary conditions. A sequence of geometrically refined meshes for the L-shape with singularity at the reentrant corner is shown in figure 2(d−e).

3. AN ERROR INDICATOR FOR THE DIRECT BEM

Consider the mixed boundary value problem

\[
\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2
\]

\[ u = g_1 \quad \text{on } \Gamma_1 \]

\[ \frac{\partial u}{\partial n} = g_2 \quad \text{on } \Gamma_2 \]

(9)

The direct boundary element method is derived from Green's formula and yields the well-known integral equations for \( u \) and \( q := \frac{\partial u}{\partial n} \).
\[ V_1(u,q) := \]
\[ c_0 u(s) - \int_{r_1} u(s) \frac{\partial}{\partial s} \log |x(s) - x(\sigma)| \, ds + \]
\[ + \int_{r_2} q(s) \log |x(s) - x(\sigma)| \, ds = \]
\[ \int_{r_1} g_1(s) \frac{\partial}{\partial s} \log |x(s) - x(\sigma)| \, ds - \int_{r_2} g_2(s) \log |x(s) - x(\sigma)| \, ds \]
\[ \text{for } \sigma \in r_2 \]

and

\[ V_2(u,q) := \]
\[ - \int_{r_1} u(s) \frac{\partial}{\partial s} \log |x(s) - x(\sigma)| \, ds + \]
\[ + \int_{r_2} q(s) \log |x(s) - x(\sigma)| \, ds = -c_0 g_1(\sigma) + \]
\[ + \int_{r_1} g_1(s) \frac{\partial}{\partial s} \log |x(s) - x(\sigma)| \, ds - \int_{r_2} g(s) \log |x(s) - x(\sigma)| \, ds \]
\[ \text{for } \sigma \in r_1 \]

Inserting \( \Sigma U_i N_i(s) \) and \( \Sigma Q_i N_i(s) \) and collocating at the points \( \sigma_i, i=1, \ldots, n \) yields the system of linear equations

\[ G \ U = H \ Q \]

for the unknowns in the vectors \( \hat{U}=(U_1, \ldots, U_n), \ Q=(Q_1, \ldots, Q_n) \).

Let now \( w=(u,q) \), \( V=(V_1, V_2) \). Then (10) can formally be written as

\[ V w = f \]
\[ \text{on } r \]

Integral equations are frequently analysed in Sobolev spaces \( H^r \) with real exponents \( r \) and the norm
\[ \| w \|_r^2 = \int \frac{(1 - |\xi|^2)^{r/2}}{r} \tilde{w}(\xi) \, d\xi \]

with the Fourier transform

\[ \tilde{w}(\xi) = \int e^{-2\pi i \xi s} w(s) \, ds \]

For integral operators of type (10) the following property is proven in \(^{20}\) under regularity assumptions to the boundary \( \Gamma \):

Theorem: (Hsiao, Wendland)

There are two constants \( C_1 \) and \( C_2 \) so that

\[ C_1 \| w \|_{t-1} \leq \| VW \|_t \leq C_2 \| w \|_{t-1} \]  \( \quad \) (12)

for \( t \in \mathbb{R} \), \( u \in H^{t-1}(\Gamma) \).

Let now \( W \) be an approximation of \( w \), then we have for the error \( e = W - w \)

\[ We = VW - VW = VW - f = r \]  \( \quad \) (13)

and

\[ \frac{1}{C_2} \| r \|_t \leq \| e \|_{t-1} \leq \frac{1}{C_1} \| r \|_t \]  \( \quad \) (14)

The residual \( r \) of the approximate solution \( W \) is a computable quantity, so (14) is of the same form as (4) with an error estimator \( \eta := \| r \|_t \) for the error \( \| e \|_{t-1} \).

Unfortunately \( C_1 \) and \( C_2 \) are not known, it is not clear, if they tend to 1 in case of mesh-refinement or even, if they can be chosen independent of the special approximation \( W \). But the numerical examples in the last chapter will show, that these estimators provide enough information to control an hp-extension process.

If the exact solution (and thus the error \( e \)) is in \( L^2(\Gamma) \), (14) reads as

\[ \frac{1}{C_2} \| r \|_{H^1} \leq \| e \|_{L^2} \leq \frac{1}{C_1} \| r \|_{H^1} \]  \( \quad \) (14')

with the \( H^1 \)-norm
\[ \| r \|^2_{H^1} := \int r^2 + \frac{\partial r}{\partial s}^2 \, ds \]

(15)

Let now \( R := \bigcup_{i=1}^{m} R_i \) with elements \( R_i \), then

\[ \eta^2 = \| r \|^2_{H^1} = \sum_{i=1}^{m} \int r^2 + \left( \frac{\partial r}{\partial s} \right)^2 ds =: \sum_{i=1}^{m} \lambda_i^2 \]

(16)

This defines error indicators \( \lambda_i \) for each element on the boundary.

4. IMPLEMENTATION OF AN HP-VERSION BEM

The 'core' of the adaptive hp-version algorithm described at the end of chapter 2 is a code which is able to compute BEM solutions with variable polynomial degree over the elements. The following shape functions, which were first introduced for the hierarchical p-version in FEM by Peano\textsuperscript{14}, are used.

\[
\begin{align*}
N_0 &= 1/2 \ (1 - \xi) \\
N_1 &= 1/2 \ (1 + \xi) \\
N_p &= 1/p! \ (\xi^p - b) \text{ for } p > 1 \\
& \quad \text{for } p > 1, \quad \text{where } b = 1 \text{ if } p \text{ is even} \\
& \quad \text{and } b = \xi \text{ if } p \text{ is odd} \\
& \quad -1 \leq \xi \leq 1
\end{align*}
\]

(17)

A major problem in the implementation of the p-version, especially for higher p-degrees, is the computation of diagonal and near diagonal elements of the matrix G in (11), i.e. integrals of the form

\[
\int_{\Gamma_i} N_p(s) \log |x(s) - x(\sigma)| \, ds
\]

(18)

where \( \sigma \in \Gamma_i \). Alarcon\textsuperscript{11} computes (18) for higher p-degrees with a numerical integration formula of Stroud-type. Yet he reports an error of nearly three percent in the solution coefficients, using polynomial degree \( p=3 \) in an example with a cubic function as exact solution. Obviously the error can only be due to the numerical integration of the influence matrices. The situation becomes much worse, if higher order shape functions are used or if the mesh is strongly graded, just the interesting case for an hp-version. Severe oscillations are observed even for Gaussian integration formula with 24 points. So it is strongly recommended to integrate (18) analytically, if \( \sigma \) is in the field.
element $r_i$. As an analytic integration for higher polynomial degrees is somewhat involved, a scientific formula manipulation program (muMATH\textsuperscript{21}) was used, yielding the integral for each $\sigma \in r_i$ for $p = 1$ to 8. Evaluation of (18) for example for $p = 6$ needs about 100 multiplications but only two evaluations of the logarithm and is thus much cheaper than a numerical integration of high order. An alternative to a fully analytic integration is described in\textsuperscript{22}, where the integrals are transformed into an analytically integrable singular part and a smooth part which is integrated numerically.

The next problem is an approximate computation of the error indicators $\lambda_i$ defined in (16). By construction, $r = 0$ at the collocation points $\sigma_i$. Choosing center points between two adjacent collocation points as 'test points', evaluating $V_1$ resp. $V_2$ at these points and computing integral (1) by a trapezoidal rule with collocation and test points gives a sufficiently accurate evaluation of the error indicators. It should be noted that the computation of these indicators is cheap; in the numerical examples the computation time was about 10-20\% of the time for establishing and solving system (11).

Adaptive h-, p- and hp-versions of the BEM can now easily be implemented. Based on the error indicators described above the extension strategies are the following:

**h-version.** If the error indicator of an element is greater than $\chi^*$ (maximal error indicator), $0 < \chi < 1$, the element is divided into two elements of equal length.

**p-version.** If the error indicator of an element is greater than $\chi^*$ (maximal error indicator), $0 < \chi < 1$, the polynomial degree of the element is increased by 1.

**hp-version.** If the error indicator of an element is greater than $\chi^*$ (maximal error indicator), $0 < \chi < 1$, then
- if the element is not adjacent to a singular point, the polynomial degree is increased by 1,
- else the element is divided into two, with a geometric progression factor $\rho$ for refinement towards the singularity. The polynomial degree is kept fix.

5. NUMERICAL EXAMPLES

In two numerical examples various extension strategies are shown. The domain for both examples is the L-shape of figure (3), with cartesian coordinates $(x,y)$ and polar coordinates $(r,\alpha)$ centered at the reentrant corner. $r_i$ is the union of all
edges, where \( u \) is prescribed, \( \Gamma_2 \) are all the edges with prescribed normal derivative. Problems similar to those presented below have been studied in \(^{23}\) using singular shape functions at the reentrant corner.

Example 1. Boundary conditions are prescribed so that the exact solution has the form

\[
\begin{align*}
\text{Example 1, Boundary conditions are prescribed so that the exact solution has the form}& \\
\frac{\partial u}{\partial n} &\text{ has a singularity of order } r^{-1/3} \text{ at the origin.}
\end{align*}
\]

\[u(r,\alpha) = r^{2/3} \sin(2/3 \alpha)\]  

(19)

First, the behaviour of the p-version for various meshes shall be investigated. Figure (4) shows the error

\[
\| U - u \|_L^2 + \| Q - q \|_L^2
\]

for \( \text{mesh}(i), i=2(2)12 \), where \( i \) indicates the number of geometric refinement steps towards the singularity. For this example, a geometric progression factor \( \rho = .5 \) was used. Note that \( \text{mesh}(12) \) has an element of length \( .5^{12} \approx 2.5 \times 10^{-4} \) at the reentrant corner. Uniform p-degree was chosen for all elements, p ranging in the example from 1 to 6.

Figure (4) shows essentially the same behaviour of the p-version which is observed in finite element applications (figure (1)), i.e. a quickly converging preasymptotic range and asymptotically a levelling off for each p-degree. The lower left envelope of the curves is bent down, showing the exponential rate of convergence for an hp-version.

Of course, the convergence depends on the geometric progression factor \( \rho \). For the FEM is could be shown in \(^3\) that the optimal factor is close to .15 yielding a very strong refinement towards the singularity. Numerical experiments with the BEM suggest a progression factor between .1 and .3. Therefore a factor of .2 will be used in all subsequent examples with the hp-version. Figure (5) shows the convergence for various uniform and adaptive extension strategies. As expected, the best possible strategy is the hp-extension, which shows the desired exponential rate of convergence. It is remarkable that the adaptive h-version with linear elements is nearly as good as the hp-version. The reason is that the exact solution is extremely smooth off from the singularity, so, except at the reentrant corner, there is nearly no refinement necessary. Table 1 shows the effectivity index \( \theta = \eta/\| \eta \|_\infty \) for all extension processes in figure (5). \( \theta \) is a measure of how good \( \eta \) estimates the true error \( \| \eta \|_\infty \). Because of the lack of theory in the development of the error estimation it is astonishing that
especially for higher polynomial degrees \( \eta \) estimates the magnitude of the exact error reasonably well.

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Table 1: Number of degrees of freedom and effectivity index for example 1

Example 2. Domain and type of boundary conditions are the same as in example 1, but now the boundary conditions are chosen so that the exact solution is

\[
u = r^{2/3} \sin(2/3 \alpha) + \sin(2x) \cosh(2y)\]  

(20)

The type of the singularity is of course exactly the same as in example 1, the exact solution is plotted over the arclength in figure (6). Figure (7) shows the convergence for various extension processes. There are some remarkable differences to example 1, which are due to the oscillatory behaviour of the exact solution in the smooth part. For a low
number of degrees of freedom, most of the error is in the smooth part of the solution. So all adaptive methods have to refine more or less uniformly to reduce this error. This explains why now the adaptive h-version yields nearly no improvement compared to the uniform h-version. For the same reason uniform p-, adaptive p- and adaptive hp-version show nearly the same exponential convergence up to about 30 degrees of freedom and an error of 10%. Beyond this point, the singularity is dominant, the p-versions level off and the hp-version shown its superiority.

Figure (8) shows the distribution of p for an adaptive p-computation with 40 degrees of freedom, figure (9) shows mesh and p-distribution for an hp-computation with 43 degrees of freedom. In figure (10 a-d) the error is plotted along the arclength, for an adaptive h-computation (72 dofs), an adaptive p-refinement with 30 dofs and p=5 at the reentrant corner, for the p-refinement shown in figure (8) and for the hp-version of figure (9). Of course, in all cases, the error at the origin is infinite. In the interior the adaptive h-version and the p-version with p=5 at the reentrant corner show significant error. The superiority of the hp-version can be seen in figures (11 a,b), which plots the solution and the error in the close vicinity of the singularity. The interval (0.1) on the arclength was chosen, i.e. one tenth of the edge adjacent to the reentrant corner. Whereas adaptive h- and adaptive p-versions oscillate significantly around the exact solution, the hp-version yields with only 43 degrees of freedom a result which is extremely accurate up to the singularity.

The effectivity index for the second example is for low polynomial degree, especially for uniform and adaptive h-version not so good as in the first example, an index of up to 4 was observed, yet for higher p-degrees again good estimation of the total error is provided.

CONCLUSIONS

Some of the recent results in adaptive finite element methods have been transformed to the boundary element method. As model problem, a collocation BEM for the direct formulation of the potential problem has been chosen. All properties which have been observed in FEM, could be shown in the numerical examples for the BEM, too. An hp-version yields very accurate results and converges exponentially even in the presence of singularities. It has been pointed out, that the integration of the influence matrices has to be performed very carefully and it was suggested to integrate the singular integrals analytically with the aid of formula manipulation programs.

Finally it should be pointed out, that there is a strong need for a rigorous mathematical foundation of a posteriori error estimations for the BEM, which could open the applications of adaptive methods for the BEM in 3 dimensions.
REFERENCES


21/ MuMATH symbolic mathematics package. Reference manual, The Soft Warehouse, P.O. Box 11174, Honolulu, Hawaii 96828, USA.
Figure 1: Convergence of a p-version FEM for different meshes

Figure 2 (a-c): FE-meshes with 0, 1 and 2 refinement layers

Figure 2 (d-f): BE-meshes with 0, 1 and 2 refinement layers
Figure 3: Domain and boundary conditions for examples 1 and 2

Figure 4: Error of p-version on different meshes for example 1

Figure 5: Error of extension processes for example 1
Figure 6: Exact solution for example 2

Figure 7: Error of extension processes for example 2

Figure 8: p-distribution for an adaptive p-computation (40 dofs)

Figure 9: Mesh and p-distribution for an adaptive hp-computation (43 dofs)
Figure 10 (a-d): Absolute error for example 2
Figure 11.2: Absolute error near reentrant corner

Figure 11.3: Solutions near reentrant corner

\begin{align*}
P &= 5^a \\
P &= 8^a \\
ADAPTIVE HP^4 \\
ADAPTIVE H^+ \\
\end{align*}