AN ADAPTIVE HP-VERSION IN THE FINITE ELEMENT METHOD

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SUMMARY

Adaptivity is now widely accepted in finite element methods. Most adaptive codes refine the finited element mesh locally controlled by some *a posteriori* estimation. In this paper an adaptive hp-version is presented. The algorithm increases the polynomial degree p and refines the finite element mesh ,i.e. decreases the local mesh-width h. Numerical examples show that even in presence of singularities in the exact solution exponential rate of convergence is obtained.

1. INTRODUCTION

There are three ways to achieve convergence in the finite element method: the h-version improves the accuracy of an approximation by refining the mesh and using shape functions of usually low degree. The p-version uses a fixed mesh but increases the polynomial degree of the shape functions to improve its accuracy and to obtain convergence. This method has been analysed during the past 5 to ten years,[1,2,3] and there has been some very promising software development[4,5] which proves the superiority of the p-version over the h-version. A combination of the h- and p-version, i.e. a simultaneous local mesh refinement and increase of the polynomial degree is called hp-version. It has been shown theoretically [2,6] and practically [7,8] that exponential rate of convergence in the energy norm can by achieved by this method even in cases with singularities in the exact solution. In a prototype finite element expert system[8] optimal combinations of mesh and polynomial degree are predicted from a starting computation with low polynomial degree on a coarse mesh. Using this prediction the user constructs with the help of this expert system a mesh-degree-combination which yields the desired accuracy at minimal computational cost.

In this paper an alternative to the expert system mentioned above will be presented. The algorithm described below is fully adaptive, i.e. starts on a coarse mesh with low polynomial degree and refines in several cycles completely automatically using *a posteriori* estimations of the distribution of the error in energy norm on every mesh. In contrast to [7,8] the polynomial degree of the shape functions needs not be constant over the entire mesh, i.e. every element can have polynomial order which is adjusted to yield the desired accuracy with minimal cost.

2. P- AND HP-VERSIONS IN FEM

As model problem, consider

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \in \mathbb{R}^2 \\ u = u_0 \quad \text{on } \Gamma_1 \quad &; \quad \frac{\partial u}{\partial n} = g_0 \quad \text{on } \Gamma_2 \end{aligned} \tag{1}$$

The smoothness of the solution u of (1) depends on the shape of the boundary and on f, u_0 and g_0. Assume first that u is analytic up to the boundary of Ω. Then the error $\|e\|_E := \|U-u\|_E$ of an approximation U to u in the p-version, i.e. an increase of the polynomial degree p of the elements on a fixed mesh converges exponentially in the energy norm.

$$\|e\|_E \leq C\, e^{-\alpha N(p)^{1/2}} \qquad p \longrightarrow \infty \tag{2}$$

C, α, are positive constants, N(p) is the number of degrees of freedom depending on the polynomial degree p. If there are reentrant corners in Ω or if there is a sudden change of the boundary condition then the exact solution can be written in the form

$$u = u_0 + \sum_{i=1}^{\infty} C_i r^{\lambda_i} g_i(\theta) \tag{3}$$

C_i are stress intensity factors, u_0 and g_i smooth functions, (r,θ) polar coordinates centered at the singularity and λ_i ordered increasingly. On a mesh as shown in figure 1a the p-version converges as in figure 2 curve (a). An exponential preassymptotic range is observed (curved down) and assymtotically the convergence is levelling off to an algebraic rate (straight line) which is governed by the power of the singularity, i.e. by λ_1. If a geometrically refined mesh as in figure 1b or 1c is used, similar S-curves can be observed (curves (b) and (c) in figure 2), yet shifted compared to that of mesh 1a. An optimal hp-version of the FEM 'switches' now from one mesh to a geometrically refined one just at the intersection points of the convergence curves always staying on the lower left envelope of the curves. This envelope itself is 'bent down', i.e. shows exponential convergence rate in the energy norm. This behaviour has been proven theoretically in [2,6] and shown numerically in [7] and [8]. Moreover it has been proven [2] that the optimal geometric progression factor for the hp-version is independent of the strength of the singularity and should be chosen as .15, yielding a very strong grading toward the singular point. Yet the optimal combination of number of refinement layers and polynomial degree depends on the stress intensity factors C_i and the exponents λ_i.

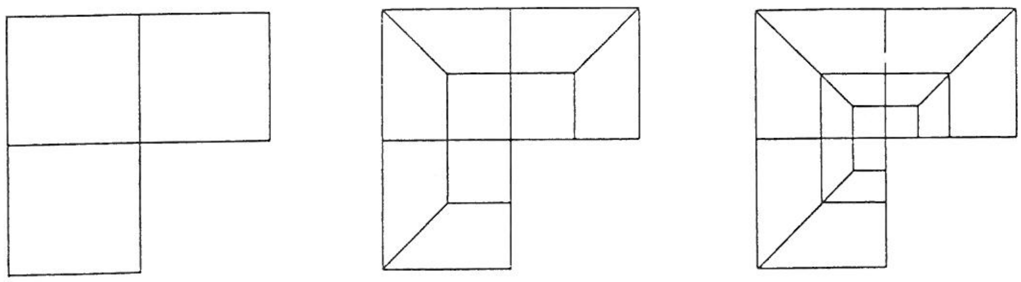

Figure 1a-c: Mesh with 0,1 and 2 refinement layers

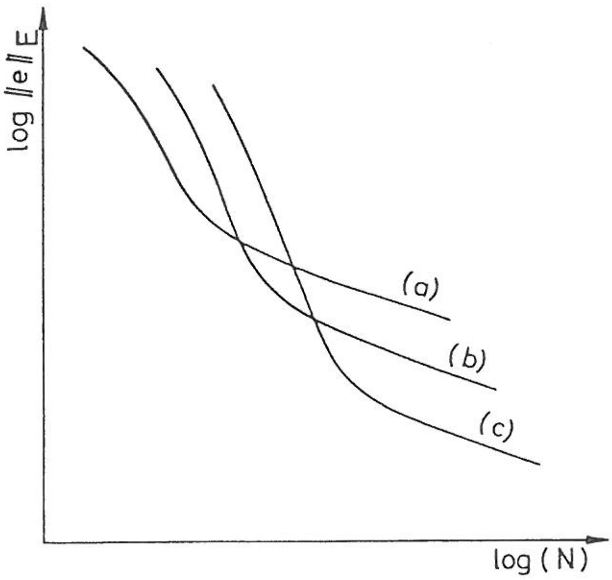

Figure 2 : Convergence of a p-version FEM for different meshes

3. AN ADAPTIVE HP-VERSION

The core of the adaptive hp-version is the p-version finite element code for linear potential, elasticity and Reissner-Mindlin-plate problems which is presented in [5]. The polynomial degree p can be varied freely over the mesh in all variables. Error indicators and estimators are similar to those presented in [9]. On the edges Γ_i of an element i the jumps $J_x(U)$ and $J_y(U)$ of the derivatives of the approximate solution U in x- and y-direction are computed and integrated to the error indicator

$$\lambda_i^2 = \frac{h}{24\,p} \int_{\Gamma_i} J_x(U)^2 + J_y(U)^2 \, d\Gamma \qquad (4)$$

h is the diameter of the element, p the polynomial degree. The error estimator for the error in energy norm is then defined as

$$\eta^2 := \sum \lambda_i^2 \qquad (5)$$

where the sum ranges over all elements.

Now an adaptive hp-version can be defined. As the goal is to achieve exponentional rate of convergence, the strategy is to increase the polynomial degree p in smooth parts of the solution and to refine geometrically at singularities. The basic algorithm has the following form:
Step 1: Choose a <u>basic mesh</u>, which is just fine enough to describe geometry, boundary conditions and loads of the problem.
Step 2 : Seperate the elements of the basic mesh into two parts, those, where the exact solution is expected to be smooth (called non-critical elements) and those adjacent to a singular point of the exact solution, e.g. reentrant corners, points of change of boundary conditions etc.(called critical elements).
Step 3: Assign polynomial degree p=1 to each element.
Step 4: Perform a FEM-computation and compute error indicators for each element.
If the accuracy estimated by the error estimator is sufficient, STOP.
Step 5: For each element decide if the error indicator is above a prespecified level, i.e. if the accuracy has to be improved. If yes, then
    for noncritical elements increase the polynomial degree by 1,
    for critical elements refine geometrically towards the singularity in this element.
Step 6: Goto step 4.

4. NUMERICAL EXAMPLES

In two numerical examples the behaviour of various extension strategies will be compared. The uniform h-version (marked as 'H 2' in the plots) refines, starting from the basic meshes uniformly and uses polynomial degree p=1 on all elements. The uniform p-version ('P 2' in the plots) uses the basic mesh and increases the p-degree uniformly over the mesh. The adaptive h-version [10] ('H 1') uses elements of degree 1 and refines locally controlled by the error indicators (4). In the adaptive p-version ('P 1') the basic mesh is unchanged but the polynomial degree is increased adaptively over the mesh, controlled again by the error indicators (4). The adaptive hp-version was run in two variants. One increases the polynomial degree uniformly over the mesh and refines locally at the singularities ('HP 2'). The other ('HP 1') varies the polynomial degree over the mesh and refines locally as defined in the algorithm of chapter 3. For the hp-versions a list of possible singularities, i.e. points of change of boundary conditions and reentrant corners was provided as input data to the program.

Example 1. As domain of computation the rectangle $\Omega = (-50,0)\times(-7,0)$ was chosen with the boundary conditions

$$u(-50,y) = u_0 \quad \text{for } -7 \le y \le 0$$

$$u(0,y) = 0 \quad \text{for } -7 \le y \le -3.5 \quad ; \quad \frac{\partial u}{\partial n} = 0 \text{ elsewhere.}$$

u_0 was chosen so that the exact solution could be computed analytically. Due to the change of boundary conditions at the point (0,-3.5) the exact solution shows a singularity of order

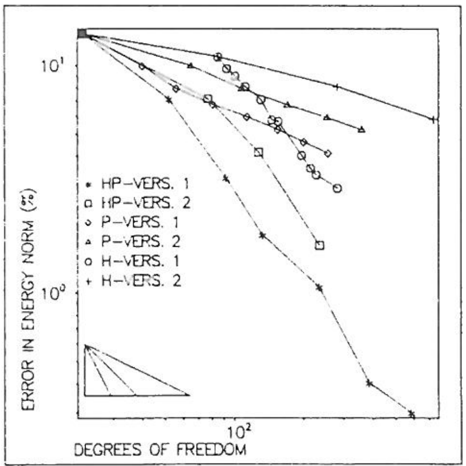

Figure 3 : Error in energy norm for example 1

$r^{-1/2}$ in the flux. On the other hand the exact solution is extremely smooth (essentially linear) in the rest of the domain.

Figure (3) shows the convergence in energy norm for the extension processes described above. The two adaptive hp-versions show superior accuracy, with only 600 degrees of freedom an error of less than .3 % is achieved. There is also a significant difference between the hp-version with uniform p-degree ('HP 2') and with variable p-degree ('HP 1'). This is due to the large smooth part of the solution where HP 1 'wastes' degrees of freedom whereas HP 2 uses only linear or quadratic elements in this part of the domain. Both HP 1 and HP 2 show the desired exponential rate of convergence in energy norm.

Example 2. The domain of computation with equipotential lines for example 2 is shown in figure (4). As the exact solution for this example is not known, the exact energy was estimated by extrapolation from very fine meshes with high polynomial degree. Figure (5) shows an adaptively refined h-version mesh for linear elements and figure (6) gives the mesh constructed by the adaptive hp-code. The different 'strengthes' of the various singularities are reflected in the different number of refinement layers at these points. Due to the strong geometric refinement towards the singularities not all refinement layers can be seen in the plots. For example at the change of boundary conditions at the lower boundary of the domain there are 5 refinement layers towards the singular point.

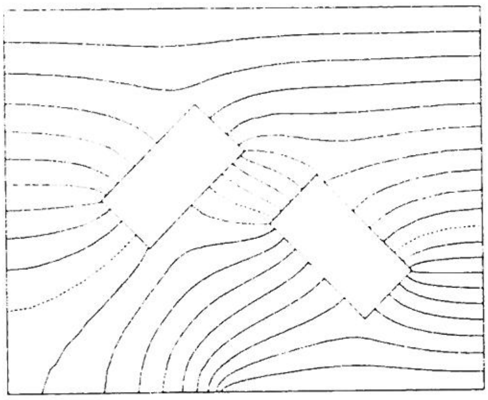

Figure 4 : Domain of computation with equipotential lines for example 2

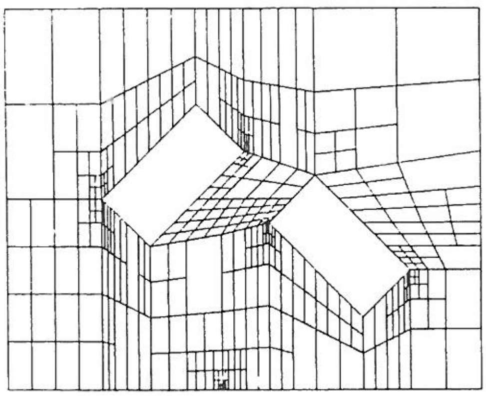

Figure 5 : Adaptively refined mesh (h-version)

In figure (7) the convergence for the extension processes is plotted. Essentially the same behaviour can be observed as in example 1, yet now there is nearly no difference between adaptive and non-adaptive p-versions and between HP 1 and HP 2. This is due to the fact that nearly the whole domain is under the influence of one of the 9 singularities leading to a nearly uniform optimal p-distribution. Again the convergence curves for the hp-versions are 'bent down' showing the exponentially decreasing error.

The effectivity index $\theta=\eta/\|e\|$, which gives a measure of the quality of the error estimator (5) is for all examples reasonably close to 1. For example HP 2 in problem 2 yielded an index of 1.17 for 38 dofs and .96 for 1104 degrees of freedom.

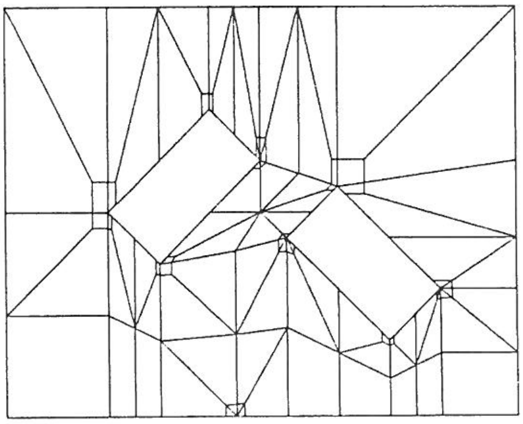

Figure 6 : Adaptively refined mesh (hp-version)

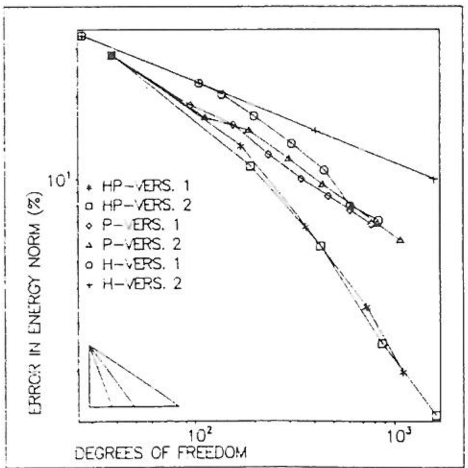

Figure 7 : Convergence in energy norm for example 2

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