A zooming-technique using a hierarchical hp-version of the finite element method

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INTRODUCTION

The hp-version of the finite element method (e.g. Guo,Babuska 1986) combines local mesh refinement with an increase of the polynomial order of the shape functions. It has been shown theoretically and in many numerical examples (e.g.Holzer et.al. 1990), that exponential rate of convergence in energy norm and excellent accuracy can be obtained for linear elliptic boundary value problems. The hp-version has also been applied successfully to more general problems like reaction diffusion equations or nonlinear Navier Stokes equations (Demkowicz et.al. 1990). Recently a variant of the hp-version as a combination of a high order approximation with a domain decomposition method has been suggested by the author (Rank 1992) . This 'hp-d-version' is similar to the 's-version' (Fish 1992) using a superposition of independent finite element meshes. The suggested approach is also related to domain decomposition methods analysed by Bramble,Ewing et.al. (1990) and to the multi-level splitting method of Yserentant (1986). The basic idea can be explained as follows. In a first step of the analysis a pure p-version approximation is performed on a coarse finite element mesh. Controlled by user interaction or by an a posteriori error estimation the coarse mesh is then covered partially by a geometrically independent fine mesh. On this second mesh a low order approximation is performed and the global approximation is defined as the *hierarchical sum* of the p-approximation on the coarse mesh and the h-approximation on the fine mesh. Global continuity of the finite element solution can be guaranteed by imposing homogeneous conditions at the fine mesh boundary. The hierarchical nature of the approximation also reflects in the structure of the arising linear equation system and can be used in an efficient solution algorithm.

In the next section algorithmic details are addressed, and it is shown how to apply this domain decomposition method for multiscale problems. The p-version is used to model the large scale solution behaviour, the h-version approximation being coupled consistently for simulation on the microscale. In the last section numerical examples are presented showing the ability of the method to efficiently and accurately model problems with a scale ratio of more than four orders of magnitude.

THE H-P DOMAIN DECOMPOSITION

As our model problem we will consider a reaction-diffusion equation

$$-\nabla(D(x)\nabla u) \; + \; ku \; = \; f \; in \; \Omega_1 \tag{1}$$

with appropriate boundary conditions. A subdomain $\Omega_2 \subset \Omega_1$ with boundary Γ_2 as in Figure 1 shall be defined and we will assume that a typical scale of Ω_2 is much smaller than the diameter of Ω_1. Ω_2 will cover local features important for an accurate finite element simulation on the global domain Ω_1. To mention a few, these features could be concentrated loads, locally inhomogeneous material , sharp reaction and shock fronts or localization zones. In the following a finite element space will be constructed allowing for a globally C^0-continuous approximation on a composite mesh scematically shown in figure 2.
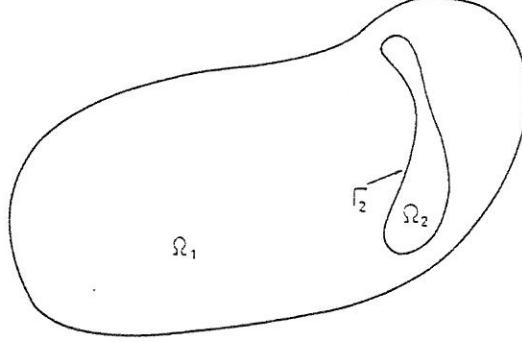

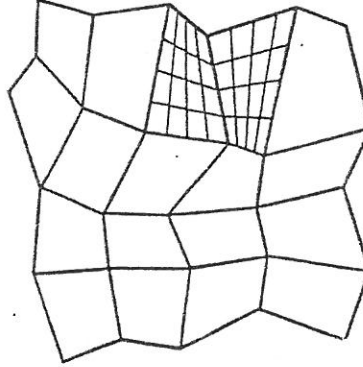Figure 1 : Domains and boundaries

Figure 2 : Composed mesh

Let $T_1 = \{t_{1,i}, i \in I_1\}$ be a regular triangulation of Ω_1 which is chosen so that there is a subset $I_{2,1} \subset I_1$ with $\Omega_2 \subset \cup\{t_{1,i}, \; i \in I_{2,1}\}$.

Let $T_2 = \{t_{2,i}, \; i \in I_2\}$ be a regular triangulation of Ω_2 which is chosen so that for every $t_{2,i}$ in T_2 there is *exactly one* j in $I_{2,1}$ with $t_{2,i} \subset t_{1,j}$ i.e. elements of T_2 do not intersect edges of the triangulation T_1.

On Ω_2 we choose now S_{h,Ω_2} as a 'standard' finite element space over T_2 and a subspace

$$S^0_{h,\Omega_2} := \{v \in S_{h,\Omega_2}|\; v = 0 \; on \; \Gamma_2\} \tag{2}$$

Typically, we use for S_{h,Ω_2} an approximation of low order finite elements, for example $p = 1$ or $p = 2$. Similary, we define a p-version approximation space S_{p,Ω_1} over the triangulation T_1 with a subspace

$$S^0_{p,\Omega_1} = S_{p,\Omega_1} \setminus S^0_{h,\Omega_2} \tag{3}$$

The global approximation space is now defined as

$$S_{h,p} = S_{h,\Omega_2}^0 \oplus S_{p,\Omega_1}^0 \tag{4}$$

i.e. every function in $S_{h,p}$ can be represented by the sum of a (global) p-version function and a (local) h-version function. To be precise let $\{N_i^h \mid i \in I_h\}$ be the set of shape functions of the h-approximation on the fine mesh, $\{N_i^p \mid i \in I_p\}$ the set of shape functions of the p-approximation on the coarse mesh. With

$$u_p = \sum_{i \in I_p} x_i^p N_i^p = \mathbf{x}^{pT} \mathbf{N}^p \ and \ u_h = \sum_{i \in I_h} x_i^h N_i^h = \mathbf{x}^{hT} \mathbf{N}^h \tag{5}$$

and real coefficient vectors $\mathbf{x}^p$ and $\mathbf{x}^h$, every $u = u_h + u_p \in S_{h,p}$ is continuous by construction. The weak formulation for a hierarchical FE-approximation of (1), using

$$B(u,v) = \int_{\Omega_1} D(x) \nabla u \nabla v + kuv \ d\Omega \quad and \quad F(v) = \int_{\Omega_1} fv \ d\Omega \tag{6}$$

as bilinear form and load functional can now be formulated.

Find $u_{FE} = u_h + u_p \in S_{h,p}$ *so that for every test function* $v_p \in S_{p,\Omega_1}^0$ *and* $v_h \in S_{h,\Omega_2}^0$

$$B(u_p + u_h, v_p) = F(v_p) \quad and \quad B(u_p + u_h, v_h) = F(v_h) \tag{7}$$

In matrix form the weak formulation can be written as

$$\mathbf{A} \ \mathbf{x} = \begin{pmatrix} A_{pp} & A_{ph} \\ A_{ph}^t & A_{hh} \end{pmatrix} \begin{pmatrix} \mathbf{x}^p \\ \mathbf{x}^h \end{pmatrix} = \begin{pmatrix} f_p \\ f_h \end{pmatrix} \tag{8}$$

with submatrices

$$A_{pp} = (a_{ij})_{i,j \in I_p} \ , \ A_{hh} = (a_{ij})_{i,j \in I_h} \ , \ A_{hp} = (a_{ij})_{i \in I_h, j \in I_p} \ , \ a_{ij} = B(N_i, N_j) \tag{9}$$
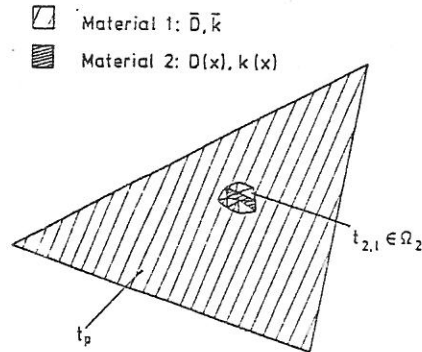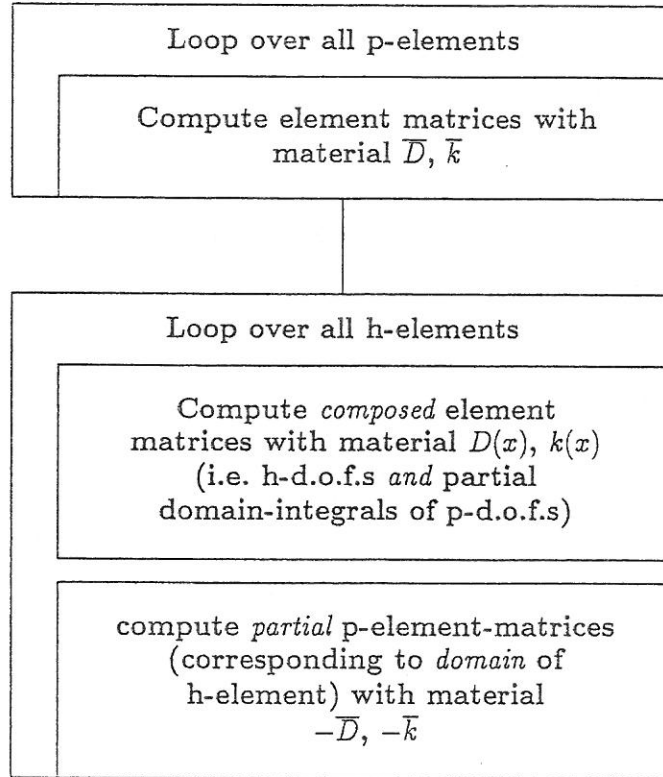

Figure 3 Element with overlay

The righthand side of (8) is defined analoguously. A question to be addressed next is the computation of element matrix and load vectors for locally inhomogeneous material. Consider a situation scematically shown in Figure 3, a single element t_p being

partially overlaid by the refined domain Ω_2. Only the shaded part in Ω_2 is assumed to have material coefficients $D(x), k(x)$, differing from 'global' material constants $\overline{D}$ and $\overline{k}$ in the rest of t_p. Let $N_i \in S_{p,\Omega_1}^0$ and $N_j \in S_{h,p}$ be functions having support in t_p. The coefficient $a_{ij}^{t_p}$ of the element matrix of t_p is given as

$$a_{ij}^{t_p} := \int_{t_p} D(x)\, \nabla N_i\, \nabla N_j\, +\, k(x)N_iN_j\, dx = \int_{t_p} \overline{D}\, \nabla N_i\, \nabla N_j\, +\overline{k}N_iN_j\, dx\, +$$

$$+ \sum_{t_{2,l} \subset \Omega_2 \cap t_p} \int_{t_{2,l}} D(x)\, \nabla N_i \nabla N_j\, +\, k(x)N_iN_j\, dx$$

$$- \sum_{t_{2,l} \subset \Omega_2 \cap t_p} \int_{t_{2,l}} \overline{D}\, \nabla N_i \nabla N_j\, +\, \overline{k}N_iN_j\, dx$$

$$(10)$$

Thus the element matrix of t_p can be computed by a *composed* integration rule taking into account the locally different material in Ω_2 in the following

Algorithm

Loop over all p-elements
Compute element matrices with material $\overline{D}, \overline{k}$

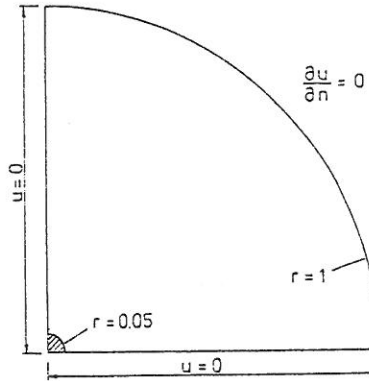| Loop over all h-elements |
| --- |
| Compute *composed* element matrices with material $D(x)$, $k(x)$ (i.e. h-d.o.f.s *and* partial domain-integrals of p-d.o.f.s) |
| compute *partial* p-element-matrices (corresponding to *domain* of h-element) with material $-\overline{D}, -\overline{k}$ |

NUMERICAL EXAMPLES

Figure 4 : domain of computation for example 1

The domain of computation of the first example is given in figure 4, showing two concentric quarter-circles with radius 1 for Ω_1 and 0.05 for Ω_2. In Ω_2 diffusion and reaction coefficient are chosen to be $D = .05$ and $k = 10000$; in $\Omega_1 \backslash \Omega_2$ $D = 1$ and $k = 0$. Homogeneous Dirichlet b.c. are set along the x- and y-axis, natural b.c. are used along the circular arc. In order to be able to compute an exact solution the righthand side of (1) is set to

$$f = -D(r)\frac{16}{r^2}\, g(r)\, \sin(4\theta)$$

$g(r)$ being the solution of an auxiliary 1D-problem

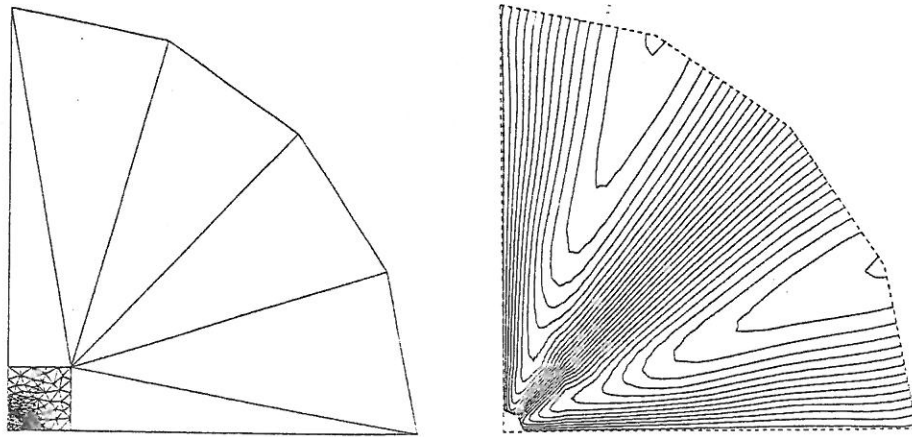$$D(r)(u_{rr} - \frac{1}{r}u_r) - k(r)u = 0 \ , \ u(0) = 0 \ , \ u(1) = 1$$

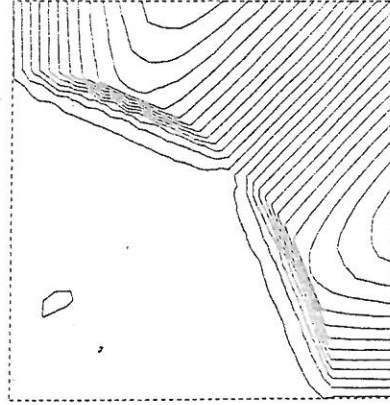Figure 5 : Composed mesh and global solution

Figure 6 : local solution

The exact solution decays like a modified Bessel function of the first kind of order 0 for $r < 0.05$ with an exponential decay length of approximately 2/1000, yielding an interface layer on a microscopic scale. Figure (5) shows the composed mesh with the (global) circularly oscillating solution. The local solution near the interface is given in Figure 6.

A computation with 370 degrees of freedom in the fine mesh and 81 degrees of freedom for polynomial degree $p = 4$ in the coarse mesh yields a pointwise error distribution as shown in Figure 7 for the global scale. The maximal error (plotted in dark) is 0.065 compared to a maximal solution value of 1. The error distribution around the interface is shown in Figure 8 with a maximal error of 0.025 compared to a maximal solution value of 0.35 in the plotted window.

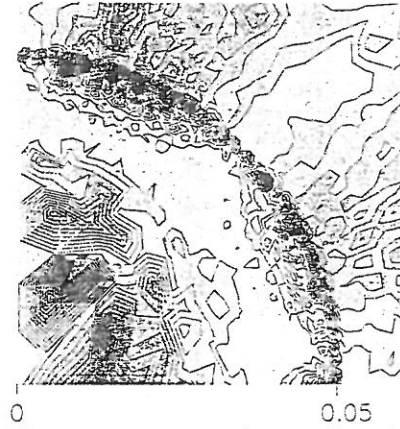Figure 7 : Error on macroscopic scale

Figure 8 : error on microscopic scale

In the second example $k = 0$ $D = 1$ everywhere, boundary conditions are set as indicated in Figure 9, showing on the lefthand side the geometrically refined mesh of the macroscopic scale and on the righthand side an enlarged area around the origin with a local heat source (plotted dark). $u = 0$ for $-1000 \leq x \leq -1$, $y = 0$ and $u = 1$ for $0 \leq y \leq 1000$, $x = 1000$.
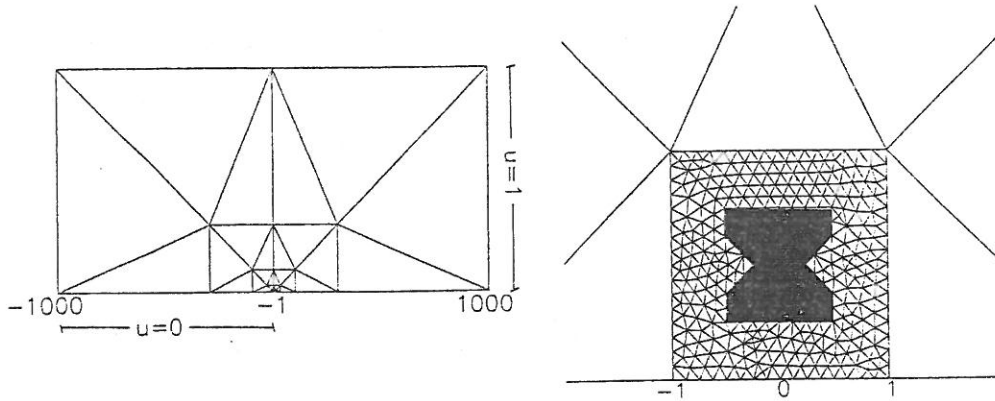

Figure 9 : Mesh, boundary conditions and local heat source for example 2

The global mesh for the p-version is refined geometrically towards the origin. A contour plot of the global solution is shown in Figure 10. Figure 11 gives the finite element solution on the microscopic scale near the origin, showing the solution at the transition from fine mesh to coarse mesh to be continuous as it is guaranteed by construction of the method. It should also be noted that this example shows well the interaction of local and global solution. Neither the microscopic nor the macroscopic behaviour could be modelled accurately without taking into account the solution on the other scale.
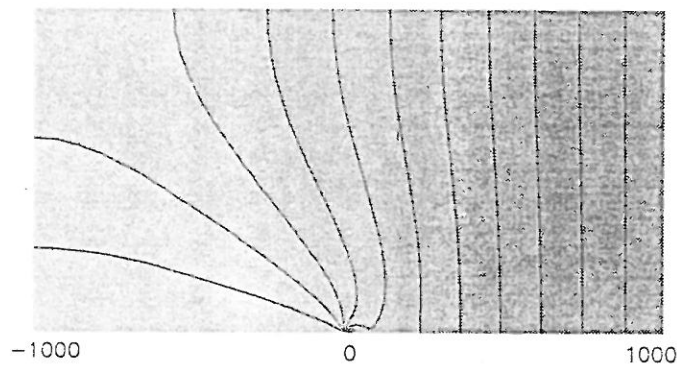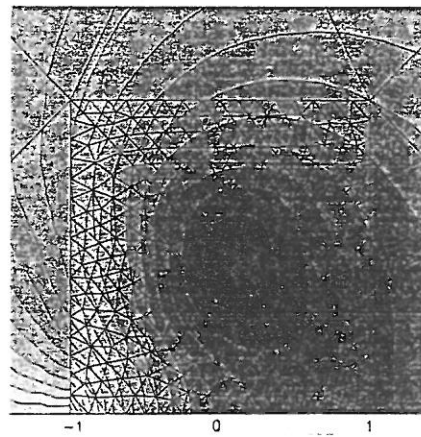
Figure 10 Solution on macroscopic scale

Figure 11 Solution near origin

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