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A MULTISCALE FINITE-ELEMENT METHOD

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Abstract—This paper describes a hierarchical overlay of a p-version finite element approximation on a coarse mesh and an h-approximation on a geometrically independent fine mesh. The length scales of the local problem may be some orders of magnitude below the scale of the global problem. Despite the incompatibility of the meshes used, continuity can easily be guaranteed in the proposed method. The paper shows how finite element meshes can be constructed adaptively on the local and the global scales. It is demonstrated how a block-iteration allows a simple and efficient implementation of the method. Typical fields of application and the efficiency of the method are shown in a numerical example. © 1997 Civil-Comp Ltd and Elsevier Science Ltd.

1. INTRODUCTION

Many problems in structural technology involve the solution of field equations being posed on domains with very different inherent length scales. To mention one class of problem, consider the computation of displacement and stress fields for the simulation of the excavation of tunnels. Near the tunnel wall, stress and strain concentrations are to be expected, often a physical or geometrical nonlinear analysis may be necessary. Therefore the field near the tunnel has to be computed accurately on a refined finite element mesh, being able to resolve geometrical details like soil layers or the reinforced structure of the tunnel walls. One major problem in these computations is the definition of boundary conditions. If the model is chosen to be too small, significant error may be caused by the introduction of an artifical outer boundary. A large model, on the other hand, will result in inacceptable computation time. Another field of application for a coupling of local global solutions are large steel structures, where geometrically complex local connections have to be analysed, whereas an external load cannot be defined directly at the connection, but has to be imposed onto the global structure. Several authors (see, e.g. [3, 4, 7]) have suggested coupling a finite element computation for the near field to a boundary element discretization for the far field of the local–global problem. As an alternative, a hierarchical coupling of a standard h-approximation for the local field with a p-approximation on the global field shall be discussed here. The basic ideas of this "hp-domain-decomposition" have been presented in Refs [8, 9]. This method is similar to the "s-version" of the finite element method developed independently by Fish [2]. In the next section the construction of the finite element approximation for the hierarchical overlay is discussed in more detail. Section 3 derives indicators for an adaptive mesh design, using also the hierarchical nature of the method. It turns out that under certain assumptions this mesh adaptation can be performed rather independently on the fine and coarse meshes.

In Section 4 the algorithmic details of the suggested approach are discussed. Besides a direct implementation as presented in Ref. [8], the method can also be implemented as an iteration between coarse and fine mesh computation in the sense of a block Gauss–Seidel-technique. Coupling terms for this formulation are strains computed on one mesh which are then imposed as loading pre-strains on the second mesh.

In Section 5 the efficiency and accuracy of the method is demonstrated by a numerical example.

2. THE HIERARCHICAL DOMAIN DECOMPOSITION

To explain the basic concepts, consider Fig. 1 with a geometrically incompatible finite element mesh. It is important, that the coarse mesh is also defined "below" the fine mesh. In our example, two (coarse) quadrilaterals are covered by the elements of the local mesh. Being precise, there exists a triangulation $T_1 = [t_{1,i},\ i \in I_1]$ of a domain Ω_1. A finite element space S_{p,Ω_1} over T_1 is defined as the set of continuous functions being piecewise polynomials of order p over each element. For the exact definition of a p-version finite element space see for example Ref. [11], for implementation details we refer to Ref. [6]. Further on we assume that there exists a subset $I_{2,1} \subset I_1$ with a domain $\Omega_2 = \cup\{t_{1,i},\ i \in I_{2,1}\}$ and define a second triangulation $T_2 = \{t_{2,i},\ i \in I_2\}$ of Ω_2 so that for every

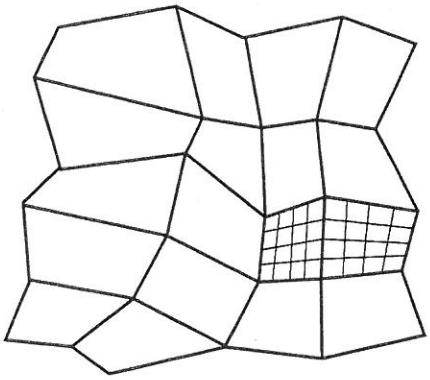

Fig. 1. Geometrically incompatible finite element mesh.

$t_{2,j} \in T_2$ there exists a $t_{1,i}$ with $t_{2,j} \subset t_{1,i}$. This condition guarantees that a *fine* element does not overlap edges of the coarse mesh.

S_{h,Ω_2} shall denote a "standard" finite element approximation using for example bilinear shape functions on T_2.

Let

$$S^0_{h,\Omega_2} := \{v \in S_{h,\Omega_2} | v = 0 \quad \text{on} \quad \Gamma_2\}. \tag{1}$$

Next we eliminate linear dependencies of the h- and the p-space defining

$$S^0_{p,\Omega_1} := S_{p_1,\Omega_1} / S^0_{h_1,\Omega_2}. \tag{2}$$

Now the finite element space of the *hierarchical overlay* is constructed by

$$S_{h,p} = S^0_{h,\Omega_2} \oplus S^0_{p,\Omega_1}. \tag{3}$$

Every element $v = v_p + v_h \in S_{h,p}$ is defined as the hierarchical sum of an h- and p-function being *continuous by construction* due to the homogeneous boundary values of the h-functions at the transition of the coarse to the fine mesh.

To set up a boundary value problem, let $B(u, v)$ be a bilinear form and $f(v)$ be a linear load functional.

In the weak form of the boundary value problem, a function u_{ex} satisfying geometric boundary conditions has to be found, so that

$$\mathbf{B}(u_{ex}, v) = f(v) \quad \forall v \in H \tag{4}$$

with an appropriate test space H.

As usual, the finite element approximation is then defined as the solution of

$$\mathbf{B}(u_{\mathrm{FE}_1}, V) = f(V) \quad \forall V \in S_{h,p}. \tag{5}$$

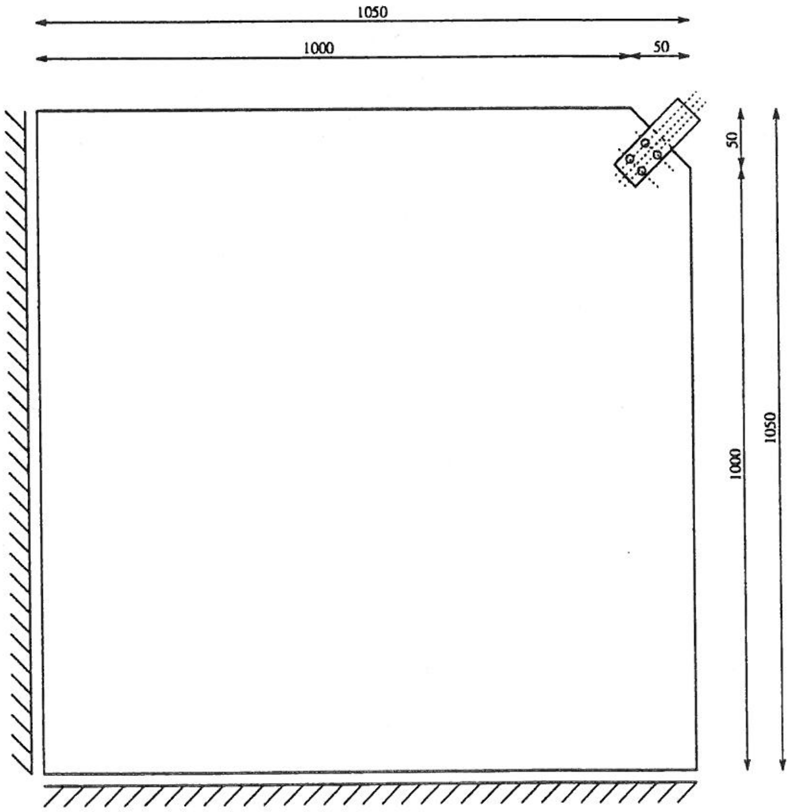

Fig. 2. Example plate.

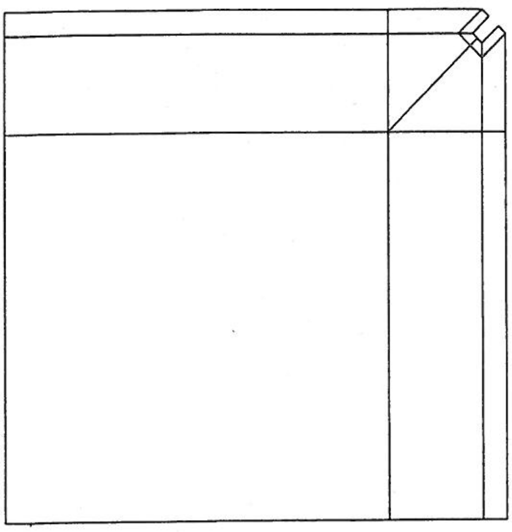

Fig. 3. Global mesh.

Using the hierarchical nature of $S_{h,p}$, eqn (5) is equivalent to the solution of the following problem:

Find $u_{FE} = u_h + u_p \in S_{h,p}$ so that for every $v_h \in S^0_{h,\Omega_2}$, $v_p \in S^0_{p,\Omega_1}$

$$\mathbf{B}(u_p + u_h, v_p) = f(v_p)$$

$$\mathbf{B}(u_p + u_h, v_h) = f(v_h). \tag{6}$$

The resulting linear equation system also reflects the hierarchical nature of this formulation

$$\begin{pmatrix} \mathbf{A}_{pp} & \mathbf{A}_{ph} \\ \mathbf{A}^{\mathrm{T}}_{ph} & \mathbf{A}_{hh} \end{pmatrix} \begin{pmatrix} x_p \\ x_h \end{pmatrix} = \begin{pmatrix} f_p \\ f_h \end{pmatrix}. \tag{7}$$

An efficient strategy to solve this equation system without an explicit computation of the coupling matrix A_{ph} will be described in Section 4.

3. ADAPTIVE MESH CONSTRUCTION

For the class of problems considered in the Introduction we can assume that an adaptive mesh refinement will be especially important in the near field, e.g. in Ω_2.

The following two questions shall be addressed in this section:

- How can the discretisation error be estimated in Ω_2?
- Is it possible to construct a mesh on Ω_2, without having to solve the coupled hierarchical system in every adaptation step?

Let us assume that a first approximation u_p has already been computed. For u_h we then get

$$\mathbf{B}(u_h, v_h) = f(v_h) - \mathbf{B}(u_p, v_h) \tag{8}$$

for every $v_h \in S^0_{h_1,\Omega_2}$. This condition is yet just a discretization of the following weakly formulated boundary value on the local domain Ω_2:

Find

$$u_1 \quad \text{on} \quad \Omega_2, \quad u_1|_{\Gamma_2} = 0$$

and

$$\mathbf{B}(u_1, v) = \tilde{f}(v) := f(v) - \mathbf{B}(u_p, v) \tag{9}$$

for all functions v of the test space.

Problem (8) is now a "standard" finite element approximation with the usual low order shape functions to the boundary value problem (9). Therefore the approximation error of (8) vs (9) can be estimated by well-known *a posteriori* procedures, e.g. by the "jump"-indicators of Babuska and Miller [1], by residual indicators of Johnson and Hansbo [5] or by stress smoothing as used in the Zienkiewicz–Zhu error estimation.

The question remains how the solution u_1 in eqn (9) on Ω_2 is related to the exact solution u_{ex} of the original problem (4).

We consider the boundary value problem (4) restricted to Ω_2.

$$\mathbf{B}(\hat{u}, v) = f(v) \quad \text{with} \quad \hat{u} = u_{ex} \quad \text{on} \quad \Gamma_2 \tag{10}$$

and define $u_2 := \hat{u} - u_p$.

Then

$$\mathbf{B}(u_2, v) = f(v) - \mathbf{B}(u_p, v) \quad \text{with} \quad u_2 = u_{ex} - u_p \quad \text{on} \quad \Gamma_2. \tag{11}$$

Comparing eqns (9) and (11) we find that u_1 satisfies instead of the exact boundary condition $u = u_{ex} - u_p$ the disturbed boundary condition $u = 0$ on Γ_2.

Thus, in an adaptive algorithm, besides the "usual" error in Ω_2 the disturbed boundary condition has to be controlled. An adaptive mesh refinement can now be performed in the following algorithm:

(1) construct initial meshes on Ω_1 and Ω_2, set $i = 0, j = 0$;
(2) solve the coupled problem for $u_h^{(i)}$, $u_p^{(j)}$;
(3) consider the *decoupled* sub-problem on Ω_2

$$\mathbf{B}(u_h, v_h) = f(v_h) - B(u_p^{(j)}, v_h) \quad \text{with} \quad u_h|_{\Gamma_2} = 0,$$

construct a sequence of h-approximations u_h in an adaptive mesh refinement on Ω_2;
(4) set $i = i + 1; j = j + 1$, and solve the coupled problem with the mesh constructed in step 3 for Ω_2 to compute $u_h^{(i)}$, $u_p^{(j)}$;
(5) if the difference of $u_p^{(j)}$ and $u_p^{(j-1)}$ on Γ_2 is small enough: STOP, else GOTO 3.

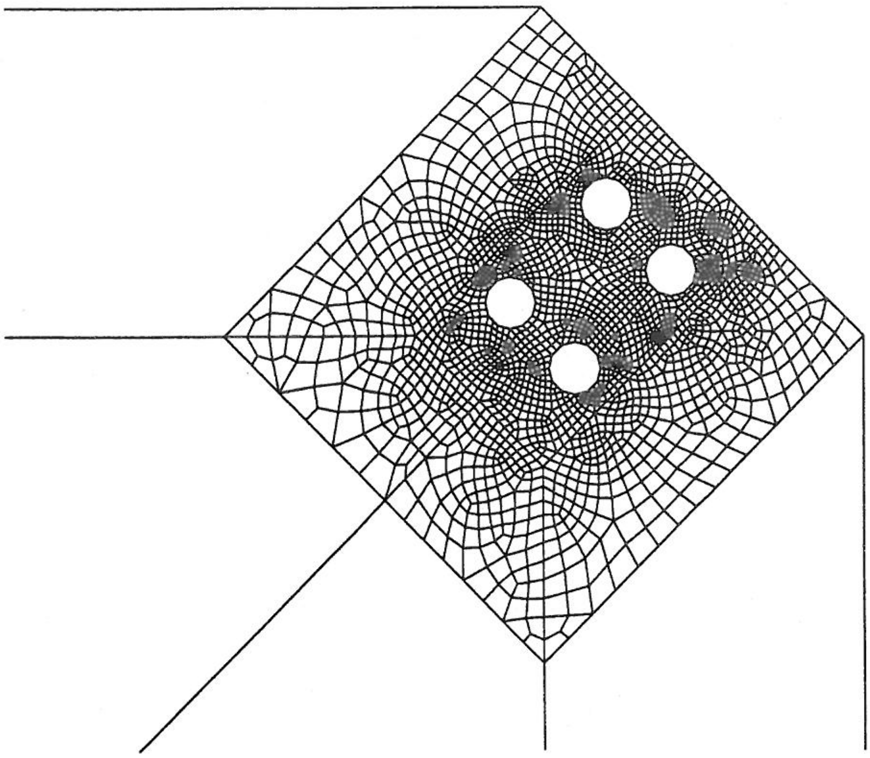

Fig. 4. Local–global mesh.

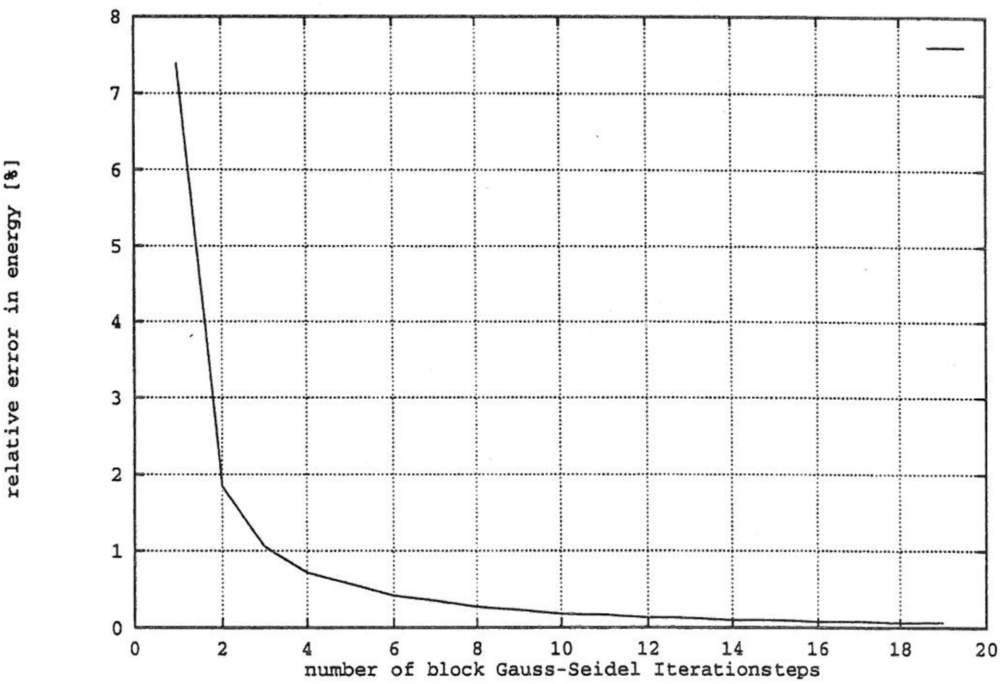

Fig. 5. Convergence of energy.

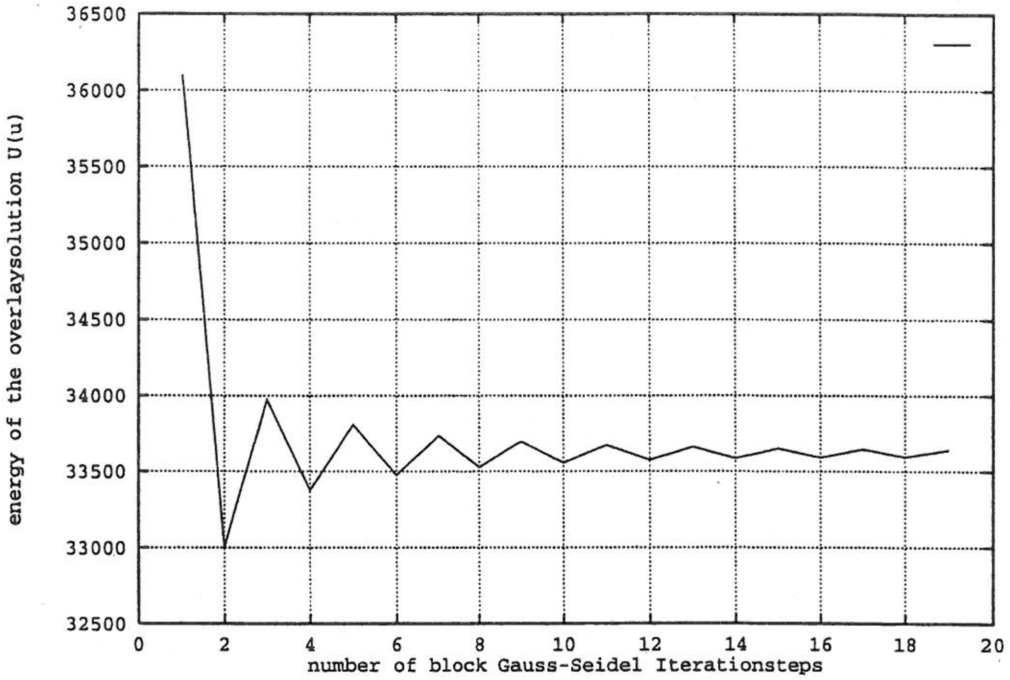

Fig. 6. Error in energy.

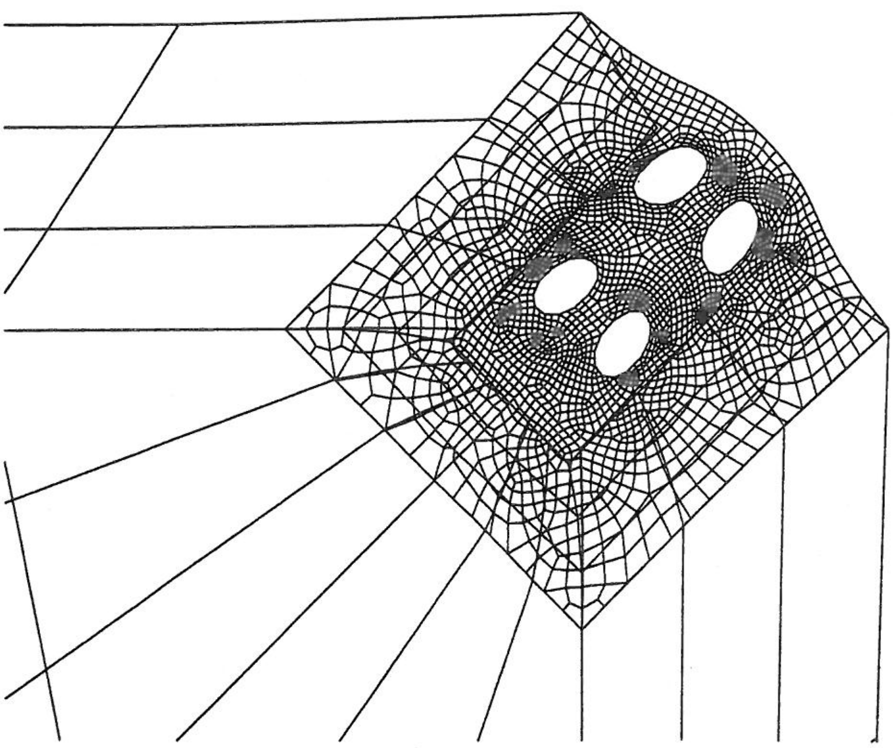

Fig. 7. Deformed structure.

4. IMPLEMENTATION

The solution of the coupled equation system 7 can be performed efficiently in a block Gauss–Seidel-iteration. The coupling matrices A_{hp} and A_{hp}^T, respectively, are brought to the righthand side of the equation systems and are considered together with the corresponding iterates as additional load terms

$$\mathbf{A}_{pp}x_p^{(n+1)} = f_p - A_{hp} \cdot x_h^{(n)}$$

$$\mathbf{A}_{hh}x_h^{(n+1)} = f_h - A_{hp}^{\mathrm{T}} \cdot x_p^{(n+1)}. \qquad (12)$$

The terms $A_{hp} \cdot x_h^{(n)}$ and $A_{hp}^{\mathrm{T}} \cdot x_p^{(n+1)}$ can be interpreted as load functionals from negative prestrains resulting from the displacement fields $x_h^{(n)}$ and $x_p^{(n+1)}$, respectively. To see this, define

$$\epsilon_0 = L \cdot \mathbf{N}_h \cdot x_h \qquad (13)$$

to be a prestrain corresponding to a given displacement field x_h, N_h being the matrix of all shape functions and L denoting the strain-operator. The discrete load vector resulting from ϵ_0 is given by

$$f(\epsilon_0) = \int_\Omega (L \cdot \mathbf{N}_p)^{\mathrm{T}} \cdot \mathbf{D} \cdot \epsilon_0 \cdot t_z \cdot \mathrm{d}\Omega \qquad (14)$$

$\mathbf{D}$ being the elasticity matrix. Inserting eqn (13) in eqn (14) and using the **B**-matrix in the standard notation as the strain operator L applied to the matrix of the shape-functions

$$f(\epsilon_0) = \int_\Omega \mathbf{B}_p^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{B}_h \cdot t_z \cdot \mathrm{d}\Omega \cdot x_h. \qquad (15)$$

We can now identify the product $\mathbf{A}_{ph} \cdot x_h$

$$\mathbf{A}_{ph} \cdot x_h = \int_\Omega B_p^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{B}_h \cdot t_z \cdot \mathrm{d}\Omega \cdot x_h. \qquad (16)$$

The importance of this observation is that it is obviously not necessary to compute the coupling matrices explicitly. Only the product with the corresponding displacement vectors is involved in the iteration (12). The consequence is, that the hierarchical local–global finite element approximation can be implemented *around* a given finite element program. The only additional program module is an interpolation of strains from one mesh to the other and the possibility to incorporate prestrains as variable loads.

5. AN EXAMPLE

Figure 2 shows a plate with a connection at the upper right edge. The load is imposed by tractions at the boundaries of the holes in the connection. The total structure is fixed at the left and bottom boundary, so the local solution, which is determined essentially by the influence of the load and the shape of the holes interacts with the global solution on a length scale being two orders of magnitude over the scale of the holes. The global finite element mesh is shown in Fig. 3, Fig. 4 giving a zoomed detail of the composed, geometrically incompatible mesh. The convergence of the block Gauss–Seidel-iteration can be seen from Figs 5 and 6, the first showing the convergence of the strain energy, the second the difference of iterated strain energies compared to the final solution. It can be seen, that only four iterations are necessary to reduce the error to less than 1%.

Finally, Fig. 7 shows the total displacement field as the sum of the p- and the h-approximation near the connection. For demonstration reasons, every p-element is devided into three by three subelements in this picture. It can clearly be seen that the total displacement is continuous at the transition of the global to the local mesh, as it is guaranteed by construction of the finite element space.

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