ON THE ACCURACY OF P-VERSION ELEMENTS FOR THE REISSNER–MINDLIN PLATE PROBLEM

ERNST RANK\(^1\)\(^*\), ROLAND KRAUSE\(^2\) AND KARIN PREUSCH\(^3\)

\(^1\) Lehrstuhl für Bauinformatik, TU München, Germany
\(^2\) The Cornell Fracture Group, Cornell University, Ithaca, U.S.A.
\(^3\) Fakultät Bauwesen, University of Dortmund, Germany

ABSTRACT

This paper addresses the question of accuracy of p-version finite element formulations for Reissner–Mindlin plate problems. Three model problems, a circular arc, a rhombic plate and a geometrically complex structure are investigated. Whereas displacements and bending moments turn out to be very accurate without any post-processing even for very coarse meshes, the quality of shear forces computed from constitutive equations is poor. It is shown that significantly improved results can be obtained, if shear forces are computed from equilibrium equations instead. A consistent computation of second derivatives of the shape functions is derived.

© 1998 John Wiley & Sons, Ltd.

KEY WORDS: Reissner–Mindlin plates; p-version; shear force computation

1. INTRODUCTION

During the past few years several commercially available new-generation finite element codes have gained considerable success in structural engineering as well as in fluid mechanics, using the concept of p-version elements which has been the topic of intensive research since the first investigations of Szabó and Peano\(^1\) in the late 1970s. There are several reasons for the attractiveness of higher-order element, the most important ones being their high accuracy, their robustness and their geometric flexibility. High accuracy is due to an exponential rate of convergence in the case of an analytic exact solution (see Reference 2) which can even be obtained for problems with singularities when an increase of the polynomial order is combined with local mesh refinement in an hp-version.\(^3\) The robustness of the p-version allows the use of strongly distorted elements\(^4\) and prevents from locking in cases of nearly incompressible material.\(^5\) Finally, the combination of p-elements and a blending function technique\(^6,7\) offers the possibility of a close coupling of the finite element analysis to geometric modelling in CAD-systems.

In this paper we will extend earlier investigations on p-elements for Reissner–Mindlin plate problems.\(^8\) Special emphasis will be laid on the computation and accuracy of moments and shear forces being often the most important quantities in engineering computations. The outline of this

\(^*\) Correspondence to: Ernst Rank, Fakultaet Bauingenieur-und vermessungswesen, U München, Arcistr 21, D80290 München, Germany. E-mail: rank@server.inf.bauwesen.tu-muenchen.de

Contract/grant sponsor: German Science Foundation DFG; Contract/grant number: RA624-1

CCC 0029–5981/98/010051–17$17.50
© 1998 John Wiley & Sons, Ltd.

Received 30 September 1996
Revised 17 November 1997
paper is the following: After setting up notation and giving remarks on our implementation of the element we will study in a first numerical example the solution quality for very coarse meshes. It will be seen that shear forces computed from constitutive equations show a considerable inaccuracy for moderately high polynomial degrees. We will then apply an idea of Stenberg and Suri\textsuperscript{9} who proved theoretically that shear forces computed from equilibrium instead, converge at higher rate. We will give some notes on practical implementation of this modified shear force computation and obtain as a main result significantly improved shear forces for our model problem. In Section 5 the rhombic plate as a standard benchmark problem will be used to compare $p$-elements and two well-known low-order elements, the Bathe–Dvorkin element DS\textsuperscript{10,11} and the DKQ element,\textsuperscript{12,13} with respect to computational effort to achieve a desired accuracy. Finally, a more complex example will show a practical application of our approach.

2. ELEMENT FORMULATION

Consider a plate $\Omega$ of thickness $t$ with mid surface in the $xy$-plane, imposed to a load $q(x, y)$ in direction $z$, as shown in Figure 1, where also moments $M_x, M_y, M_{xy} = M_{yx}$ and shear forces $Q_x, Q_y$ are depicted. Primary variables for the Reissner–Mindlin plate problem are deflection $w$ in $z$-direction and rotation $\phi_x, \phi_y$ about $x$- and $y$-axis, respectively. Strains are defined as

$$
\varepsilon = [\varepsilon_B^T, \varepsilon_S^T]^T = \begin{bmatrix}
\frac{\partial \phi_x}{\partial x} z, \frac{\partial \phi_y}{\partial y} z, \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) z, \phi_x + \frac{\partial w}{\partial x}, \phi_y + \frac{\partial w}{\partial y}
\end{bmatrix}^T
$$

(1)

where $\varepsilon_B$ and $\varepsilon_S$ denote strain components due to bending and shear respectively. The plate is in equilibrium, if

$$
\begin{align*}
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0 \\
\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y &= 0 \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - q(x, y) &= 0
\end{align*}
$$

(2)

The relation between the stress resultants $M$ and strains $\varepsilon$ is given by

$$
M = [M_x, M_y, M_{xy}]^T = \frac{t^3}{24} D_B \varepsilon_B
$$

(3)

$$
Q = [Q_x, Q_y]^T = \frac{t}{2} D_S \varepsilon_S
$$

with

$$
D_B = \frac{E}{(1-v^2)} \begin{bmatrix}
1 & v & 0 \\
v & 1 & 0 \\
0 & 0 & \frac{(1-v)}{2}
\end{bmatrix}
$$

(4)

$$
D_S = \frac{E}{2(1+v)} \begin{bmatrix}
k & 0 \\
0 & k
\end{bmatrix}
$$

(5)

and $k$ being the shear correction factor which is often set to $k = \frac{5}{6}\textsuperscript{14}$. 

At the boundary $\Gamma$ tractions $M_n, M_t, M_{nt}$ and $Q_n$ can be defined. Further it is possible to prescribe soft simple support ($w = 0$), hard simple support ($w = \phi_t = 0$) and clamped boundary conditions ($w = \phi_n = \phi_t = 0$), $\phi_n$ and $\phi_t$ being rotations in normal and tangential direction, respectively.

The weak formulation of problem (2) then reads

Find the function $u = [\phi_x^{(e)}, \phi_y^{(e)}, w^{(e)}]^T \in \mathcal{H}$ satisfying geometric boundary conditions and

$$\mathcal{B}(u, v) = \mathcal{F}(v)$$

(6)

for all test functions $v = [\phi_x^{(e)}, \phi_y^{(e)}, w^{(e)}]^T \in \mathcal{H}^0$, with appropriate Sobolev spaces $\mathcal{H}, \mathcal{H}^0$.

The bilinear form $\mathcal{B}$ is defined as sum of a bending and a shear part

$$\mathcal{B}(u, v) := \mathcal{B}_B(u, v) + \mathcal{B}_S(u, v)$$

$$:= \frac{t^3}{12} \int_{\Omega} \varepsilon^{(u)}_B : D \varepsilon^{(u)}_B \, dx \, dy + t \int_{\Omega} \varepsilon^{(u)}_S : D \varepsilon^{(u)}_S \, dx \, dy$$

(7)

and the load functional is

$$\mathcal{F}(v) := \int_{\Omega} q w^{(e)} \, dx \, dy + \int_{\Gamma} (M_n \phi_n^{(e)} + M_t \phi_t^{(e)} - Q_n w^{(e)}) \, ds$$

(8)

As usual, the finite element approximation to $u$ will be found from a subspace $S \subset \mathcal{H}$ so that (6) holds for every $V \in S^0 \subset \mathcal{H}^0$.

Our $p$-version implementation uses hierarchical basis functions on quadrilaterals with an ansatz space $A^p$ suggested by Szabó and Babuška,7 spanned by all monomials $\zeta^i \eta^j$, $i, j = 0, \ldots, p$; $i + j = 0, \ldots, p$. These monomials are supplemented for $p = 1$ by the monomial $\zeta \eta$ and for $p \geq 2$ by $\zeta^p \eta$ and $\zeta \eta^p$.

The geometric shape of the elements used in our investigations is defined by a blending function technique, being described e.g. in Reference 7. Straight lines, circular arcs or cubic splines can be used as the elements boundary shapes.
3. A FIRST ACCURACY CONSIDERATION

Our first example will study the model problem sketched in Figure 2, a quarter of a circle with a circular hole, soft simply supported on all boundaries of the domain. The goal of this example is to show the ability of p-elements to handle shapes strongly deviating from squares and to investigate the influence of boundary layers onto the solution in the interior. Bending moments and shear forces are computed directly, without smoothing, from the displacement variables $w, \phi_x, \phi_y$ by (1) and (3). Figure 3 shows three finite element meshes, mesh 3(a) consisting of only one element, 3(b) having one element layer along the boundaries and mesh 3(c) having two layers, with a ratio of the small diameters of layer elements being 1:7. Note the extreme aspect ratio of the element in mesh 3(c) closest to the boundary of about 1:75.

Plate 1 shows the vertical displacement $w$, Plate 2 moment $M_x$ and Plate 3 shear force $Q_x$ for the three meshes used, $p$ being uniformly equal to 4, 6, 8. In Figure 4 the displacement is plotted along the section A–A, and Figures 5 and 6 depict the bending moment $M_y$ and shear force $Q_x$ along the same section. Finally, Tables I and II give the computed values of $w$ and $M_y$ at points P1, P2, and P3 in Figure 2. The error in energy norm estimated from an extrapolated solution on mesh 3 for $p = 7, 8, 9$ is shown in Figure 7.

Using the solution for $p = 8$ on mesh 3 as a reference ‘exact’ solution (with an error in strain energy of 0.8 per cent estimated from extrapolation), the following observations can be made:

1. Displacement $w$ is reasonably accurate even for the one-element mesh already for $p = 6$.
   Considering point P3 near the plates centre a relative error of 4.5 per cent is well within the range of required accuracy for many engineering applications.

Figure 2. First test example with section line and test points

Figure 3. Finite element meshes for the circular plate problem

© 1998 John Wiley & Sons, Ltd.

(2) A similar observation can be made for $M_y$ (and for the moments $M_x$ and $M_{xy}$ likewise). It is especially remarkable, that the one-element solution with $p = 6$ and 8 shows at all test points an error of less than 7 per cent demonstrating the robustness of the $p$-version for even strongly distorted elements.
(3) A completely different behaviour is observed concerning the accuracy of shear force $Q_t$. The strong boundary layers which can be seen in the solution of mesh 3 for $p=8$ seem to affect the solution so strongly, that the ($p=4$)-solution is completely useless and even $p=6$ as well as $p=8$ on mesh 1 show strongly oscillating (and inaccurate) behaviour.
4. IMPROVED SHEAR FORCE COMPUTATION

The rather disappointing results for shear forces computed directly from strains give rise to the investigation of possible improvements. In Reference 8 a smoothing technique was suggested, in
Table I. Results of displacement \( w \) at evaluation points \( \cdot 10^{-3} \) (m)

<table>
<thead>
<tr>
<th></th>
<th>Point 1</th>
<th></th>
<th>Point 2</th>
<th></th>
<th>Point 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh 1</td>
<td>0.285</td>
<td>0.339</td>
<td>0.345</td>
<td>4.345</td>
<td>6.128</td>
</tr>
<tr>
<td>Mesh 2</td>
<td>0.269</td>
<td>0.318</td>
<td>0.319</td>
<td>7.020</td>
<td>7.209</td>
</tr>
<tr>
<td>Mesh 3</td>
<td>0.305</td>
<td>0.319</td>
<td>0.319</td>
<td>7.119</td>
<td>7.220</td>
</tr>
</tbody>
</table>

Table II. Results of moment \( M_y \) at evaluation points (kNm/m)

<table>
<thead>
<tr>
<th></th>
<th>Point 1</th>
<th></th>
<th>Point 2</th>
<th></th>
<th>Point 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh 1</td>
<td>-16.22</td>
<td>-34.00</td>
<td>-35.68</td>
<td>14.68</td>
<td>22.06</td>
</tr>
<tr>
<td>Mesh 2</td>
<td>-34.06</td>
<td>-32.54</td>
<td>-32.55</td>
<td>22.88</td>
<td>23.45</td>
</tr>
<tr>
<td>Mesh 3</td>
<td>-31.01</td>
<td>-32.24</td>
<td>-32.28</td>
<td>22.99</td>
<td>23.72</td>
</tr>
</tbody>
</table>

Figure 7. Convergence in energy norm

This paper we will consider improved shear force computation from equilibrium conditions (2). In an early paper,\(^\text{15}\) it was shown that for low-order elements the accuracy of shear forces computed from equilibrium, i.e. satisfaction of the first two of equations (2) in a weak sense, improves significantly. It is clear, that a direct computation of shear forces from (2) is not possible (or at least not reasonable) for low order elements, as it involves second derivatives of shape functions, being zero or constant for linear or bilinear elements. Using yet \( p \)-extension, we can follow a suggestion of Stenberg and Suri,\(^\text{9}\) who proved that the strong solution of (2) for the shear forces \( Q_x, Q_y \) improves the convergence of the \( L_2 \)-error by half an order. Although this was only shown for clamped boundary conditions and parallelogram element shape, we will demonstrate a significantly improved accuracy for general element shapes as well.

To resolve (2) for the shear forces \( Q_x \) and \( Q_y \), the second derivatives \( N_{xx}, N_{yy}, N_{xy} \) of all shape functions \( N \) have to be computed.

\( \text{© 1998 John Wiley & Sons, Ltd.} \)

Let

\[ X = (x, y)^T, \quad R = (r, s)^T \]  

be global and local co-ordinates of a point inside an element, where the second derivatives

\[ G := \frac{d^2 N}{dX^2} := \begin{pmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{pmatrix} \]  

with respect to global co-ordinates shall be computed.

Calculation of derivatives

\[ \frac{d^2 N}{dR^2} := \begin{pmatrix} N_{rr} & N_{rs} \\ N_{sr} & N_{ss} \end{pmatrix} \]  

with respect to local co-ordinates from the definition of the shape functions is straightforward. The Jacobian of the geometric mapping

\[ J := \begin{pmatrix} x_r & y_r \\ x_s & y_s \end{pmatrix} \]  

and the second derivatives

\[ \frac{d^2 x}{dR^2} = \begin{pmatrix} x_{rr} & x_{rs} \\ x_{sr} & x_{ss} \end{pmatrix} \quad \text{and} \quad \frac{d^2 y}{dR^2} = \begin{pmatrix} y_{rr} & y_{rs} \\ y_{sr} & y_{ss} \end{pmatrix} \]  

can easily be computed from the geometric mapping from local to global co-ordinates.

Then by differentiation

\[ \frac{d^2 N}{dR^2} = J \frac{d^2 N}{dX^2} J^T + N_x \frac{d^2 x}{dR^2} + N_y \frac{d^2 y}{dR^2} \]  

and resolving for the global derivatives

\[ \frac{d^2 N}{dX^2} = J^{-1} \left( \frac{d^2 N}{dR^2} - N_x \frac{d^2 x}{dR^2} - N_y \frac{d^2 y}{dR^2} \right) J^{-1 \ast} \]  

shows the influence of the element geometry in \( G \) which could only be neglected for slightly distorted elements.

Reconsidering now the numerical example of Section 3 a significant improvement of the accuracy of shear forces is seen. Oscillations due to boundary layers are strongly reduced and a computation with \( p = 6 \) even on the coarsest one-element mesh gives rather satisfactory results in the interior of the domain (see Plate 4 and Figure 8). If yet a resolution of boundary layers is desired, mesh refinement along the boundary as in meshes 2 or 3 and polynomial degree \( p = 6 \) or more is necessary. (See also the detailed investigation of boundary layer resolution of Schwab.)

### 5. THE RHOMBIC PLATE PROBLEM

The second test example is a well-known benchmark test for plate elements, being extensively studied in References 17 and 18. Figure 9 shows a rhombic plate with an evaluation point \( \textbf{M} \) at the centre and at the quarter point \( \textbf{P}_4 \) of the diagonal \( \textbf{A} - \textbf{C} \). The plate of thickness \( t = \frac{1}{50} = 0.02 \)

© 1998 John Wiley & Sons, Ltd.  
Figure 8. Shear force $Q_x$ along section line A–A (computed from equilibrium conditions)

is uniformly loaded by $p = 1.0$, Youngs modulus is set to $E = 1.0$ and Poisson's ratio is $v = 0.3$. The plate is soft and simply supported along all boundaries. $P$-version elements are used on three uniform meshes with 1, 4 and 9 elements, denoted as Mesh es 1, 2 and 3. Results for displacements $w$ and bending moments $M_I$ and $M_{II}$ are compared with those of the 'discrete
Kirchhoff element$^{12,13}$ and the element presented in References 10 and 11 being denoted as DKQ and DS4. These widely used elements were among those with best performance in the benchmark investigation of Reference 17. Concerning $p$-extension, we used for this test example a polynomial degree for rotations of one less than that for displacements, reducing the total number of degrees of freedom without strongly affecting the accuracy.

In Figures 10 and 11 convergence of displacements in the two evaluation points is plotted. $P$-extension is performed on all meshes ranging from polynomial degree $p = 4$–11 for displacements and from $p = 3$–10 for rotations. It is worth noting that the exact displacement is obviously well above the results for DS4 and DKQ on the fine meshes with about 1880 degrees of freedom (and of the analytic Kirchhoff solution plotted in the figures). This is evident from the convergence of the $p$-version on the 9-element mesh. Even on the coarsest mesh with only one element and 172 degrees of freedom a solution is obtained, which is by far superior to that of the classical elements at a comparably small number of unknowns.

Considering now moments at the plates centre $M$ as shown in Figures 12 and 13 it is observed first, that again the one-element mesh yields astonishingly accurate results. Yet the 4-element extension shows significant oscillations which have to be explained. The reason is the strong singularity of the exact solution at the oblique corner which is ‘reflected’ to the plates midpoint,

Plate 1. Displacement $w$ of Mesh 1 with $p = 4$ (left), Mesh 2 with $p = 6$ (center) and Mesh 3 with $p = 8$ (right).

Mesh 1:

- $p = 4$

Mesh 2:

- $p = 6$

Mesh 3:

- $p = 8$

Plate 2. Moment $M_x$.

© 1998 John Wiley & Sons, Ltd.

Plate 3. Shear force $Q_a$ (computed from strains)
Plate 5. Displacement $w$ of Mesh 1 and Mesh 2 with $p = 8$

Plate 6. Bending moment $M_y$ of Mesh 1 and Mesh 2 with $p = 8$
being opposite and in the same element as the corner, when the plate is meshed into four elements. As expected, the oscillations disappear in the \( p \)-extension on the 9-element mesh where now a very smooth convergence and a high accuracy with a low number of degrees of freedom is observed. Shear forces could not be compared for this example, as they are not given in the benchmark test.\(^{17} \) Results for moments at point P4 (see Reference 19) are similar and are omitted here for sake of space.

In addition to the previously obtained results for uniform \( h \)-meshes we also compare the performance of the \( p \)-version to the DS4-element in an adaptive \( h \)-version for the rhombic plate problem. Based on Reference 20 a residual-type error estimator and error indicators are employed to define a series of adaptively refined meshes. Starting on a uniform mesh with 88 elements and 287 degrees of freedom a sequence of meshes with 1080, 2878 and 5830 degrees of freedom was derived. The last mesh in this series with 1887 elements is depicted in Figure 14. In Figure 15 the
convergence of displacement $w$ at point M is plotted for the series of adaptively refined meshes, for a uniform $h$-refinement and for the $p$-extension on mesh 3. As expected, the adaptive $h$-extension is superior to the uniform $h$-refinement, but it can clearly be seen that the $p$-extension converges significantly faster than both $h$-extensions with the low-order element considered.

It is, of course, not possible to compare efficiency of low- and high-order elements only under the aspect of number of degrees of freedom necessary to obtain a certain accuracy. Naturally, setting up element matrices is much more costly for high-order elements. It is yet often not considered that the solution of the global equation system may be performed much more efficiently for high-order elements. Using, for example, $p = 8$ in the ansatz space $A^p$ defined in Section 2, about half of all shape functions are bubble modes, which can be eliminated locally. This condensation can be viewed as an efficient domain decomposition preconditioning of the global equation system (see, Reference 21), yielding an overall computational cost per degree of freedom being comparable to that of low order elements.

To confirm this observation, execution times and iteration counts for $p$-extensions on uniform $3 \times 3$ and $12 \times 12$ meshes for the rhombic plate problem are compared to a computation on a
refined mesh with more than 9000 elements and DS4-elements. This low-order computation was performed using a classical conjugate gradient solver with incomplete Cholesky preconditioning. For the $p$-extension, the conjugate gradient solver was preconditioned by diagonal scaling, after elimination of all element bubble modes. All computations were performed on a HP735/125-workstation. Table III shows the number of degrees of freedom, the number of iterations, time for equation solution and total execution time. Elimination of bubble modes, i.e. the major step of preconditioning the $p$-version is not included in equation solution time, as it is performed completely on element level during pre- and post-processing. Note that the number of iterations in the $p$-extension only grows very slowly with the $p$-degree, and that the relative amount of work is shifted drastically from equation solution to the element level with increasing $p$, showing that this $p$-version implementation would be very well suited for a parallel implementation.

6. A COMPLEX EXAMPLE

As stated in the introduction, one significant advantage of $p$-version finite element analysis is the ease of modelling a structure to obtain results of sufficient engineering accuracy. This shall be demonstrated on the example shown schematically in Figure 16, where geometry and support is
REISSNER-MINDLIN PLATE PROBLEM

\[
q = 0.025 \quad [\text{MN/m}^2]
\]

\[
E = 30000 \quad [\text{MN/m}^2]
\]

\[
u = 0.3
\]

\[
t = 0.25 \quad [\text{m}]
\]

Figure 16. Third test example

(a) Mesh 1 with 14 elements  (b) Mesh 2 with 129 elements

Figure 17. Two finite element meshes for third test example

sketched. Along all bold lines of the plate structure soft simple support is defined, at the remaining boundary parts natural boundary conditions are chosen. The plate is loaded by a uniform area load of 0.025 MN/m². Figure 17(a) shows a coarse mesh of only 14 quadrilateral elements and in Figure 17(b) a mesh refined at reentrant corners and points of change of boundary condition with 129 elements is plotted. Plate 5 shows contour lines of vertical displacements for a solution with \( p = 8 \) on the coarse mesh (left) and, as reference, on the fine mesh (right). Obviously, the accuracy of the coarse mesh solution is very well within the range of engineering necessity. The same observation can be made for results of bending moments \( M_x \) (and for the other moments likewise), being compared in the plot of not smoothed contour line in Plate 6. A more qualitative comparison can be made in Figure 18 where \( M_x \) is plotted along the section line X–X (see Figure 16). Using as a reference solution \( p = 8 \) on the refined mesh, it can be seen that only small errors are observed for \( p = 8 \) on the coarse mesh, whereas results for \( p = 4 \) are obviously not of sufficient accuracy.

Finally, in Figure 19 shear force $Q_x$ computed from equilibrium is plotted along section line $Y-Y$. Again, the accuracy of the solution for $p=8$ on the coarse mesh is satisfactory, whereas $p=4$ does not even indicate an increase of the shear force near the boundaries.

7. CONCLUSIONS

In extensive numerical investigations the accuracy of $p$-version Reissner–Mindlin plate elements was studied. It was shown that a computation of shear forces from equilibrium instead of a direct calculation from constitutive equations improves the quality significantly and yields an element...
with excellent robustness. Perhaps the most astonishing result of this investigation is that in all our test examples an accuracy, which is commonly accepted to be sufficient for engineering needs, can be obtained even on the coarsest mesh representing the geometry, if polynomial degree \( p = 6 \) or more is used. If, however, singularities or boundary layers are to be resolved in detail, a combination of high polynomial degree and local mesh refinement should be used. Taking into consideration the high geometrical flexibility of \( p \)-elements, an overall performance is observed which is superior to that of well-known low-order elements in many applications.

ACKNOWLEDGEMENTS

The work of the second author was supported by the German Science Foundation DFG under contract RA624-1.

REFERENCES