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August 1999

in:
ECCM '99, European Conference on Computational Mechanics, München, Germany, CD-ROM 1999

ECCM '99
European Conference on
Computational Mechanics
August 31 – September 3
München, Germany

A COMBINATION OF AN h– AND A p–VERSION OF THE FINITE ELEMENT METHOD FOR ELASTIC–PLASTIC PROBLEMS

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Key words: multiscale problems, elastic–plastic problems, *hp-d*–method

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Abstract. *The paper describes a hierarchical overlay of a p–version finite element approximation on a coarse mesh and an h–approximation on a geometrically independent fine mesh for elastic–plastic problems. The length scales of the local, physically nonlinear problem may be some orders of magnitude below the scale of the global, linear problem. Despite the incompatibility of the meshes used, continuity can easily be guaranteed in the proposed method. We will show in this paper how the resulting global nonlinear equation system can be solved by an iteration using a Block Gauss–Seidel scheme combined with a modified Newton–Raphson method. It is demonstrated that the bearing capacity of a slope under vertical loading significantly depends on the size of the computational domain which can be discretized with the presented method resulting in only a few additional degrees of freedom.*

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1 Introduction

The numerical treatment of structural problems is often difficult if the solution is affected by phenomena on different length scales. In simulating these multiscale problems it is essential to find a discretization that equally reflects all aspects of the problem with a sufficient accuracy. As a typical example consider the computation of the load bearing capacity of a slope under a local loading. One precondition for an accurate simulation of this problem is the adequate choice of the numerical model, i.e. the definition of the domain of computation with its boundary conditions as well as the consideration of the local nonlinear behaviour of the soil. If the domain of computation is chosen to be too small, significant errors may be caused by the introduction of an artificial outer boundary. A large model, on the other hand, may result in inacceptable computation time. The *hp-d*–method, which was presented for linear problems earlier [1],[2], will be extended here to locally elastic–plastic behaviour and is intended as an alternative to well–known strategies coupling a local finite element computation to a boundary element computation for the exterior problem [4],[5]. Our method combines the *p*–method for the discretization of the large–scale problem with an *h*–method for the local solution. Thus, the advantageous properties of both standard procedures may be optimally exploited.

2 The hierarchical domain decomposition

To explain the basic idea of the *hp-d*–method, consider the example of a strip footing near a slope under vertical loading (Figure 1). The nonlinear behaviour of the material is assumed to be elastic–perfectly–plastic, see section 3. Two meshes, a base mesh consisting of only 8 *p*–elements on Ω_b and an overlay mesh with 523 eight–noded elements on Ω_o, are chosen to discretize the slope (see Figure 2). Thus, the domain Ω is splitted into two overlapping subdomains $\Omega = \Omega_b \cup \Omega_o$. The finite element solution $\mathbf{u}_{FE}$ is now defined as the hierarchical sum of a *p*–version approximation on the base mesh $\mathbf{u}_b$ and an *h*–version approximation on the overlay mesh $\mathbf{u}_o$. C^0–continuity for $\mathbf{u}_{FE}$ can be guaranteed just by imposing homogenous boundary conditions on Γ_{bo} for $\mathbf{u}_b$ and on Γ_{ob} for $\mathbf{u}_o$.

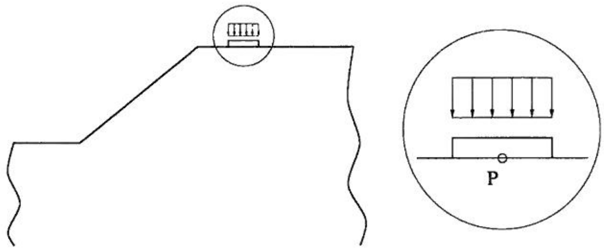

Figure 1: Strip footing near a slope under vertical loading

The decomposition of the domain Ω is chosen such that plastic strains only occur in the subdomain $\Omega_o \setminus \Omega_b$, i.e. in that part of the structure in which only an *h*–approximation is

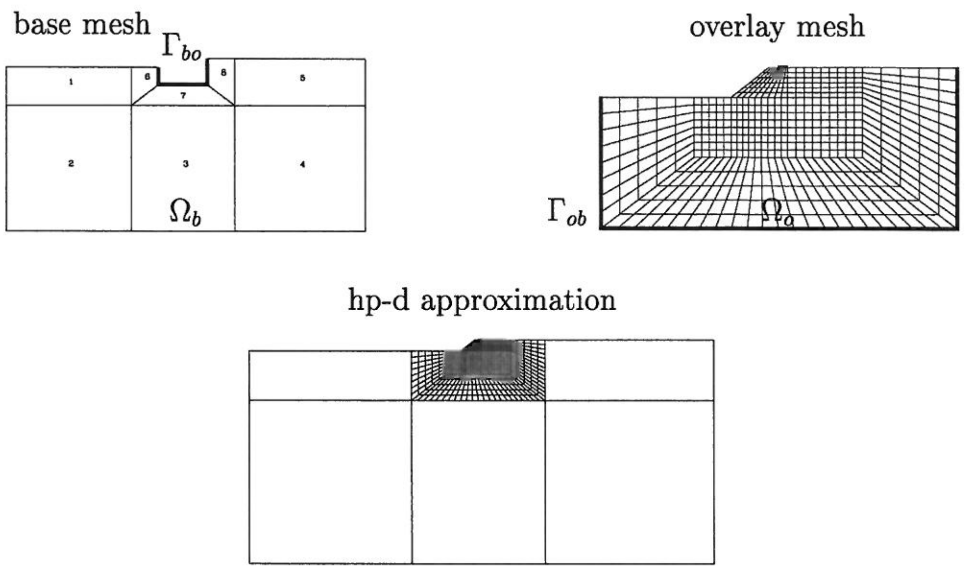

Figure 2: Hierarchical domain decomposition with partial overlay of base and overlay mesh.

performed. Due to this domain decomposition the governing nonlinear equation system exhibits the following structure

$$\begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bo} \\ \mathbf{K}_{bo}^T & {}^{(n+1)}\mathbf{K}_{oo}(\mathbf{u}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_b \\ \Delta\mathbf{u}_o \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ {}^{(n+1)}\mathbf{f}_o - {}^{(n+1)}\mathbf{g}_o(\mathbf{u}) \end{bmatrix} \tag{1}$$

with the submatrices $\mathbf{K}_{bb}, \mathbf{K}_{bo}$ reflecting linear elastic behaviour whereas the nonlinear behaviour of the structure is considered by the submatrix $\mathbf{K}_{oo}(\mathbf{u})$. ${}^{(n+1)}\mathbf{f}_o$ is the load vector of the $(n+1)$–th load step being only applied on the domain Ω_o and ${}^{(n+1)}\mathbf{g}_o(\mathbf{u})$ are the internal forces which have to be in equilibrium with the external forces. The superscript $(n+1)$ denotes the sequence of load steps being applied for an incremental solution of the nonlinear problem. The finite element solution for one load step is given by the hierarchical sum

$$\Delta\mathbf{u}_{FE} = \Delta\mathbf{u}_b + \Delta\mathbf{u}_o \tag{2}$$

of the solution $\Delta\mathbf{u}_b$ on the base mesh and the solution on the overlay mesh $\Delta\mathbf{u}_o$.

We will consider now two different strategies to solving the nonlinear equation system (1). The first strategy will be refered to as the 'direct' solution. Therefore the static condensation of $\Delta\mathbf{u}_b$ is applied to the system (1) resulting in the Schur–complement

$$\left[{}^{(n+1)}\mathbf{K}_{oo}(\mathbf{u}) - \mathbf{K}_{bo}^T \mathbf{K}_{bb}^{-1} \mathbf{K}_{bo}\right] \Delta\mathbf{u}_o = {}^{(n+1)}\mathbf{f}_o - {}^{(n+1)}\mathbf{g}_o(\mathbf{u}) \tag{3}$$

which may be solved in each load step with a Newton–Raphson iteration scheme. Note that $\Delta\mathbf{u}_b$ has to be condensed only once. The resulting nonlinear equation system (3) is therefore reduced in size resulting in a significant acceleration of the solution phase of the overall system (1).

Although this 'direct' solution of system (1) would be a very efficient strategy, we have chosen a Block Gauss–Seidel iteration which can be written as

$$\begin{array}{rcrcrcl} \mathbf{K}_{bb}\,\Delta\mathbf{u}_b^{(i+1)} & = & & & & - & \mathbf{K}_{bo}\,\Delta\mathbf{u}_o^{(i)} \\ {}^{(n+1)}\mathbf{K}_{oo}\,\Delta\mathbf{u}_o^{(i+1)} & = & {}^{(n+1)}\mathbf{f}_o & - & {}^{(n+1)}\mathbf{g}_o(\mathbf{u}) & - & \mathbf{K}_{bo}^T\,\Delta\mathbf{u}_b^{(i+1)} \end{array} \quad . \tag{4}$$

The benefit of these iteration scheme can be seen by studying the coupling terms, i.e.

$$\mathbf{K}_{bo}\,\Delta\mathbf{u}_o^{(i)} = \int_{\Omega_b\cap\Omega_o} \mathbf{B}_b^T\,\mathbf{C}\,\mathbf{B}_o\,d\Omega\,\Delta\mathbf{u}_o^{(i)} = \int_{\Omega_b\cap\Omega_o} \mathbf{B}_b^T\,\mathbf{C}\,\boldsymbol{\varepsilon}_o^{(i)}\,d\Omega\,, \tag{5}$$

$\mathbf{C}$ being the elasticity matrix and $\mathbf{B}_o$,$\mathbf{B}_b$ the $\mathbf{B}$–matrices in the standard notation as the strain operator $\mathbf{L}$ applied to the matrices of the shape functions $\mathbf{N}_b$,$\mathbf{N}_o$ of the base and the overlay mesh, respectively. The coupling term (5) can therefore be interpreted as a load functional from negative prestrains resulting from the displacement field $\Delta\mathbf{u}_o^{(i)}$. Because of the domain decomposition, where plastic strains can occur only in the domain $\Omega_o \setminus \Omega_b$ the so–called prestrains in equation (5) are pure elastic. After having found a solution with the Block Gauss–Seidel iteration, the local nonlinear behaviour of the structure on the domain $\Omega_o \setminus \Omega_b$ can be considered with a Newton–Raphson scheme.

The advantage of this second iteration scheme is that two different FEM–codes may be coupled easily. For our investigation one program [2] was used to compute the p–version solution on the base mesh whereas another one [3] was employed to consider the local nonlinear behaviour of the computational domain. The overall solution technique is illustrated in Figure 3.

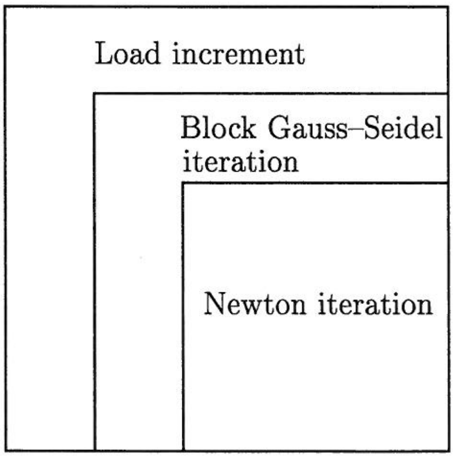

Figure 3: Solution of the nonlinear equation system

3 Formulation of the elastoplastic material model

The domain of the structure is divided into the two parts Ω_b (base mesh) and Ω_o (overlay mesh) with a common transmission zone $\Omega_b \cap \Omega_o$. Plastic flow is restricted to occur only in $\Omega_o \backslash \Omega_b$. The nonlinear material behaviour is computed by the FEM–code ISARES [3].

The nonlinear material model is based on small deformations. Hence, the linearized kinematic relations are given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right) . \tag{6}$$

The strain can be splitted into an elastic and a plastic part

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{e} + \dot{\boldsymbol{\varepsilon}}^{p} , \tag{7}$$

where the dot denote strain rates according to the applied Prandtl–Reuß plasticity model. The plastic deformations are defined by an associated flow rule in the form

$$\dot{\boldsymbol{\varepsilon}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} . \tag{8}$$

Considering the consistency condition

$$\mathrm{d}f = \frac{\partial f}{\partial \boldsymbol{\sigma}}:\mathrm{d}\boldsymbol{\sigma} = 0 , \tag{9}$$

the plastic multiplier λ is given by

$$\dot{\lambda} = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}:\boldsymbol{\sigma}}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}:\mathbf{C}:\dfrac{\partial f}{\partial \boldsymbol{\sigma}}} \tag{10}$$

in which $\mathbf{C}$ denotes the elastic material tensor. A modified hyperbolic Drucker–Prager flow function was used in order to get rid of numerical difficulties in the vicinity of the apex of the flow cone during the stress integration algorithm, see Fig. 4. Therefore, the direction of plastic flow remains well defined. The corresponding flow function reads

$$f = \left(J_2^n + \sigma_{ms}^{2n}\right)^{\frac{1}{2n}} + \alpha_f I_1 - k \leq 0 . \tag{11}$$

The first invariant of the stress tensor and the second invariant of the stress deviator tensor are given by

$$\begin{aligned} I_1 &= \mathrm{tr}[\boldsymbol{\sigma}] &&= \sigma_{ij}\,\delta_{ij} , \\ J_2 &= \tfrac{1}{2}\mathrm{tr}\big[[\mathrm{dev}\boldsymbol{\sigma}][\mathrm{dev}\boldsymbol{\sigma}]\big] &&= \tfrac{1}{2}\,\sigma_{ij}\,\sigma_{ij} . \end{aligned} \tag{12}$$

The parameters α_f, k, n and σ_{ms} are used to determine the shape of the flow surface. For the stress integration algorithm a forward Euler scheme with subincrementation was applied.

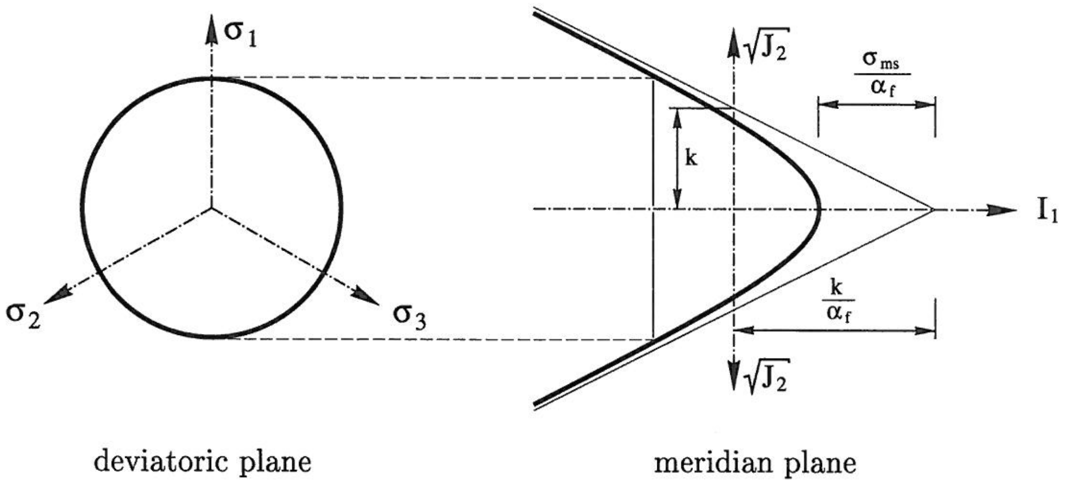

Figure 4: Graphical representation of the hyperbolic flow function

4 A numerical example

As a numerical example, consider the computation of the load bearing capacity of a slope under a local loading (Figure 1). Three different discretizations will be explored: a typical h–version discretization with 523 eight–noded quadrilateral elements (Figure 5) and two *hp-d*–version meshes varying only in the size of the base mesh, consisting of 8 p–elements with $p = 4$ (Figure 6 and 7). The overlay mesh of the two *hp-d* discretizations coincides with the h–version mesh. The displacements normal to the artificial boundary of the system are suppressed for all three discretizations. Additionally kinematic suppressions are formulated at the boundaries Γ_{bo} and Γ_{ob} where the displacements are fixed in order to obtain C^0–continuity of the approximation $\mathbf{u}_{FE}$. The number of degrees of freedom for the h–version mesh is 3130 whereas the number of additional degrees of freedom from the p–elements is 129. The number and size of load increments coincides for all computations. The material constants are as follows:

Modulus of elasticity (strip footing)	E_F	=	300000.0
Modulus of elasticity (soil)	E_B	=	30000.0
Poisson's ratio (strip footing, soil)	ν	=	0.2
friction angle/cohesion $\phi = 15/c = 5 \rightarrow$	α_f	=	0.0917
	k	=	5.134
parameter for yield criterion	σ_{ms}	=	2.0
parameter for yield criterion	n	=	4.0

Figure 8 shows the load–settlement–curves of point P, located under the foundation (Figure 1), for all three discretizations. The load bearing capacity computed by the *hp-d*–discretizations is up to 6 % higher than the one computed with the pure h–version. Comparing the solutions of the *hp-d*–version we observe that the *hp-d*–version mesh 1

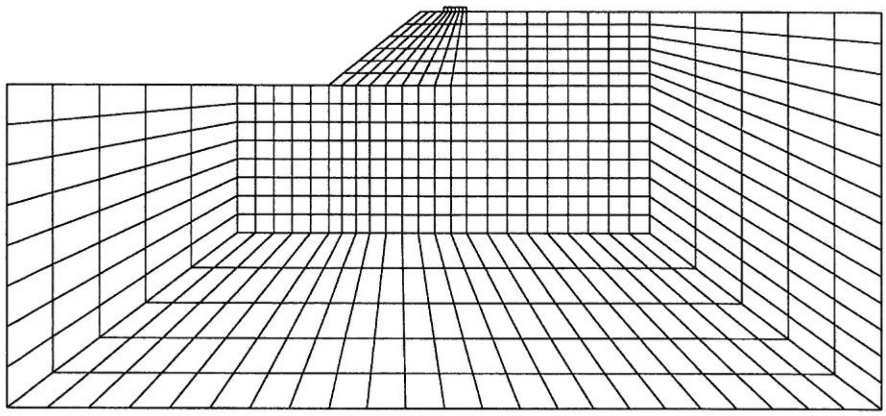

Figure 5: h–version mesh: 523 eight–noded elements with 3219 dof

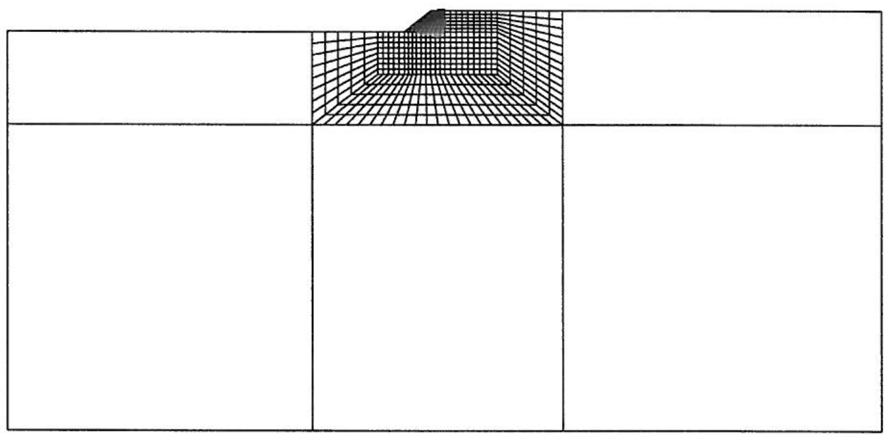

Figure 6: *hp-d*–version mesh 1

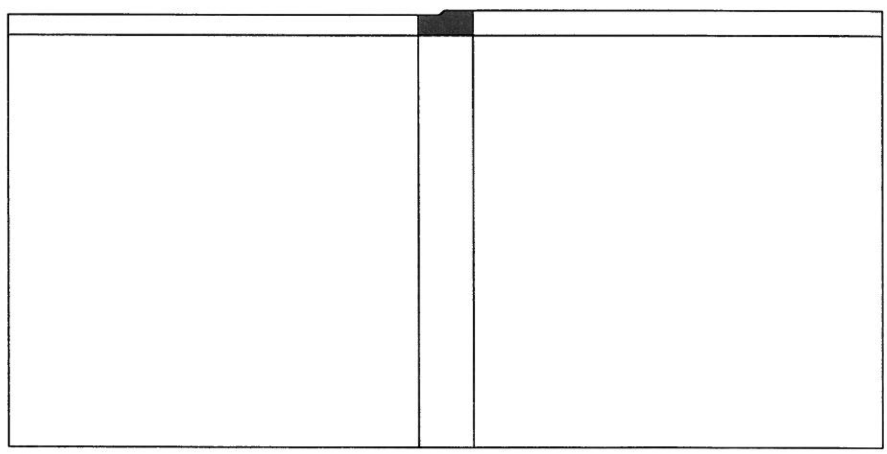

Figure 7: *hp-d*–version mesh 2

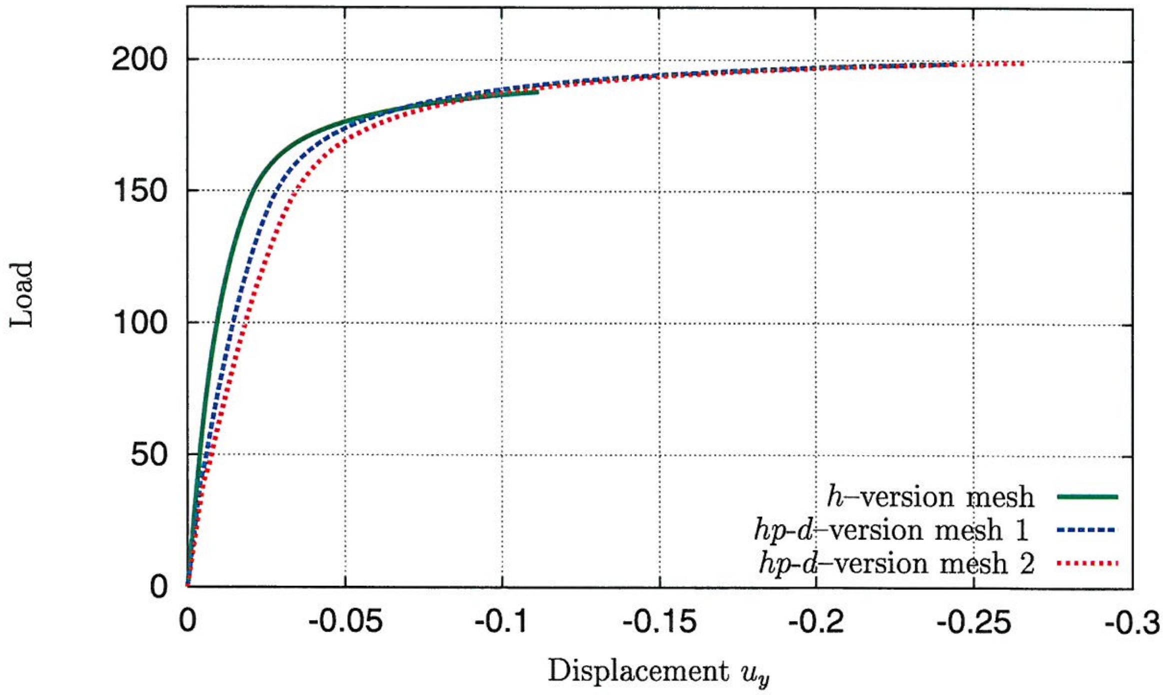

Figure 8: Load-settlement-curves of point P

(Figure 6), discretizing the smaller global domain, supplies the smaller load bearing capacity. This is due to the fact that the artificial, stiffening boundary conditions are imposed closer to the local structure.

In Figure 9, 10 and 11 the plastic zones as a result of the three computations for F=187.5 are plotted. The shape of the plastic zone is very similar for all three discretizations.

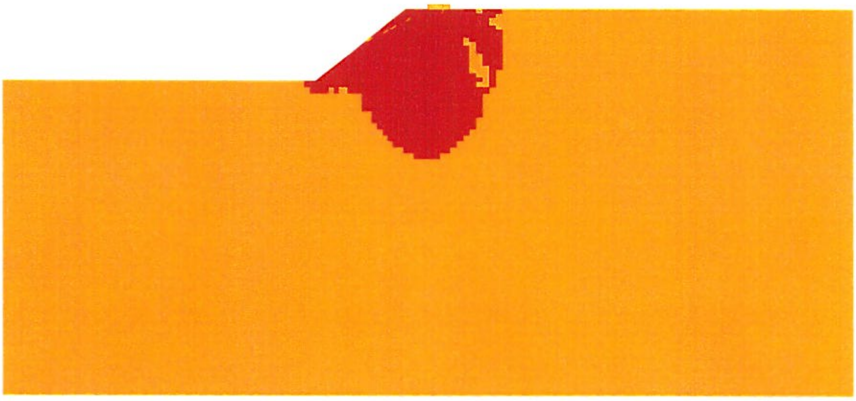

Figure 9: h–version mesh 1: plastic zone (F=187.5)

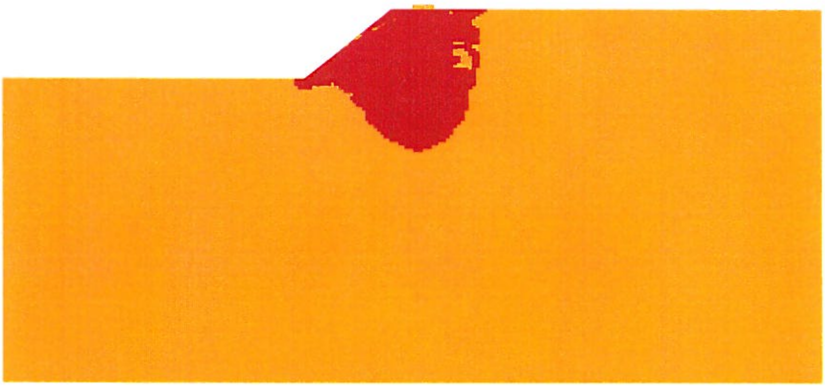

Figure 10: hp-d–version mesh 1: plastic zone (F=187.5)

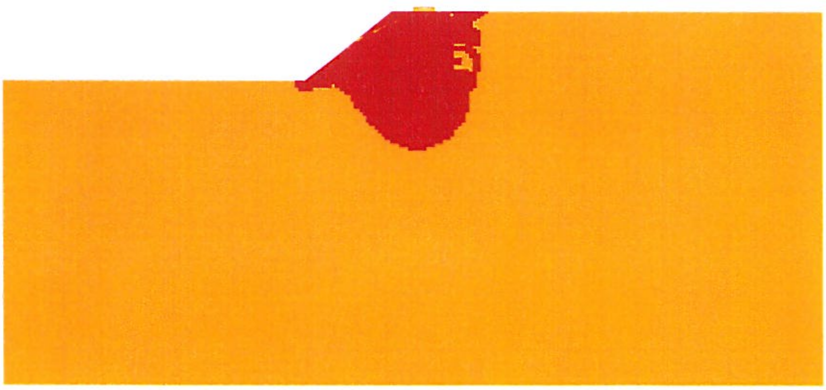

Figure 11: hp-d–version mesh 2: plastic zone (F=187.5)

5 Conclusions

The aim of this paper is to show that the *hp-d*–version of the finite element method is a suitable strategy for solving nonlinear problems with strongly different length scales. The numerical example demonstrates that the elastic domain of the slope may be discretized with only a few additional degrees of freedom whereas the local nonlinear problem may be treated as detailed as necessary due to the geometrically independent overlay mesh.

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