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Abstract. In this paper we describe a new approach to hierarchically split a three-dimensional finite element approximation into a part which can be computed by a dimensionally reduced model and a second, strictly three-dimensional approximation to local disturbances of the solution. Using the p-version, this concept can be used as a tool for a dimensional adaptivity in structural computations.
1 Introduction

The numerical analysis of thin-walled structures like plates or shells is usually performed by dimensionally reduced models like the Reissner-Mindlin-plate theory or corresponding shell formulations. Although the resulting elements are very well established, they naturally lead to difficulties and approximation errors in zones where the a priori assumptions of the theory are violated, e.g. in areas of transition from thin to thick-walled constructions or in parts of the domain where the structure is exposed to a local load. To handle this problem in practice, transition elements to couple e.g. plates to solids are frequently introduced. Stein et al. ([1]) go one step further and investigate a concept of dimensional adaptivity defining modelling error indicators which identify those parts of the structure where a model of higher dimension should be used. We present in this contribution a different approach for a consistent coupling of a dimensionally reduced model with a fully three-dimensional simulation. Earlier investigations [4] to split a two-dimensional problem into a local and a global part, where we used the p-version to approximate the global solution behaviour being overlaid by a local h-version approximation to capture local features of the problem are extended in this paper.

To outline the basic idea, we consider a middle surface $\Omega$ of a structure $\Omega \times [-t/2, t/2]$ and a subset $\Omega_o \subset \Omega$, demanding only that the kinematic assumptions of the Reissner-Mindlin theory are satisfied in $(\Omega \setminus \Omega_o) \times [-t/2, t/2]$. $\Omega_o$ may be that part of the domain, where the dimensionally reduced model is not valid e.g. the vicinity of local support or an area near a local load acting on the surface (and not on the middle surface) of the structure.

We use now a hierarchical decomposition of a (three-dimensional) finite element approximation $u_{FE}$ into two parts

\[ u_{FE} = \overline{u}_b + u_o, \]

where $\overline{u}_b$ is defined on $\Omega \times [-t/2, t/2]$, whereas $u_o$ is zero everywhere on $(\Omega \setminus \Omega_o) \times [-t/2, t/2]$. In our actual approximation, we do not compute $\overline{u}_b$ directly, but define it as a three-dimensional extension of a Reissner-Mindlin solution $u_b$ over $\Omega$, using the kinematic assumptions of this theory. $u_o$, on the other hand, is a true 3D approximation correcting $\overline{u}_b$. Requiring $u_o$ to fulfill homogeneous boundary conditions on the boundary of $\Omega_o \times [-t/2, t/2]$, $u_{FE}$ is continuous over the whole domain of computation, i.e. a conforming three-dimensional finite element approximation.

The implementation of this local correction method can readily be realized by a block iteration scheme, and we use a p-version finite element program to compute the (global) Reissner-Mindlin approximation as well as the (local) 3D-correction. To automatically detect that part $\Omega_o$ where dimensional adaptivity should be applied local Dirichlet-type problems taking advantage of the hierarchical nature of the p-version approach can be used.

We will show in numerical examples, that the outlined procedure allows an efficient three-dimensional analysis of structures, where the assumptions of dimensionally reduced models are violated locally. In those parts of the domain, where the assumptions of the Reissner-Mindlin
model are fulfilled, only the computationally more efficient 2D-approximation has to be performed.

2 A combination of the h- and p-version FEM

In [3] it was demonstrated, that three-dimensional high-order anisotropic polynomial spaces are very well suited for the accurate computation of thin-walled structures. Although this approach was shown to be by far superior to classical h-version methods with low-order elements, the computation of thin-walled structures with three-dimensional anisotropic p-elements is - with respect to CPU-time - still quite demanding. The CPU-time for a specific three-dimensional example was found to be approximately one order of magnitude higher than the time needed for corresponding two-dimensional computations. Of course, the fast growth of computer speed will overcome this problem and three-dimensional computations of thin-walled structures will become more common. Nevertheless, two-dimensional computations being less CPU demanding will be still of interest in future.

In this Section we will present a modification of a method which was presented earlier to consistently couple local and global structural computations ([4]). Here, we extend this idea to a coupling of two- and three-dimensional structural models. Using this procedure, it is possible to perform a fast dimensionally reduced analysis of thin-walled structures and to improve the approximation in areas, where significant deviations from the exact three-dimensional solution occur. After a short introduction into the basic concept it will be explained how this method can be applied to couple dimensionally reduced models with a locally full three-dimensional analysis.

2.1 Local-global coupling

In Figure 1 the basic idea of the hp-d method, which was first proposed in [5] is explained for a one dimensional model problem. Suppose that a finite element approximation on a domain $\Omega$

\[
\begin{align*}
\text{base mesh: } u_b & \quad + \quad \text{overlay mesh: } u_o \\
= & \quad \text{hp-d approximation, } C^0\text{-continuity: } u_{FE}
\end{align*}
\]

Figure 1: Basic idea of the $hp$-d method for one-dimensional problems
consists of two parts: \( u_b \) is defined as a standard \( h - \) or \( p - \) approximation on a so-called base mesh, discretizing the whole computational domain. In order to locally improve this solution - e.g. in a region where a significant error occurs - a fine overlay mesh is superimposed to the base mesh. On this overlay mesh a second part of the approximation, \( u_o \), is defined and the overall approximation \( u_{\text{FE}} \) is composed of the hierarchical sum \( u_{\text{FE}} = u_b + u_o \). The \( C^0 \)-continuity of \( u_{\text{FE}} \) is readily taken care of by imposing homogenous boundary conditions on the overlay approximation. The same procedure can now be applied to problems in two or three dimensions.

The two-dimensional situation is sketched in Figure 2. \( \Omega_1 \) with boundary \( \Gamma_1 \) is the domain of computation, being meshed in triangular or quadrilateral elements. A not necessarily connected subset of these elements is now defined as the domain \( \Omega_2 \) with boundary \( \Gamma_2 \). To be more precise,

Figure 2: The \( hp-d \) method: domain decomposition and definition of boundary conditions

we now set up the basic formulation of the \( hp-d \) method. Suppose that

\[
B(u, v) = F(v)
\]  

(1)

describes the weak form of equilibrium of a two-dimensional elasticity problem. The bilinear form is then defined as

\[
B(u, v) := \int_{\Omega} (L v)^T C (L u) t_z \, dx \, dy
\]  

(2)

where \( L \) is the standard strain-displacement operator, \( C \) is the elasticity matrix and \( t_z \) corresponds to the thickness of the two-dimensional structure. The load functional

\[
F(v) := \int_{\Omega_1} (b_x v_x + b_y v_y) t_z \, dx \, dy + \int_{\Gamma_1} (\tilde{T}_n v_n + \tilde{T}_t v_t) t_z \, ds
\]  

(3)

considers volume loads \( b = (b_x, b_y)^T \) and tractions, where \( \tilde{T}_n \) and \( \tilde{T}_t \) are normal and tangential components of the traction vector.

In order to find a finite element solution to the weak formulation (1) by applying the \( hp-d \) method, we define again the hierarchical sum

\[
u_{\text{FE}} = u_b + u_o, \quad \Gamma_1 : u_b = \bar{u}, \quad \Gamma_2 : u_o = 0,
\]  

(4)
where $u_b$ is the approximation corresponding to the discretization of the domain $\Omega_1$ and $u_o$ corresponds to the domain $\Omega_2$. On $\Gamma_1$ the essential system boundary conditions are imposed, whereas $u_o = 0$ on $\Gamma_2$ guarantees the overall $C^0$-continuity of $u_{FE}$. Inserting (4) in (1) we obtain

$$B(u_b + u_o, v_b) = F(v_b)$$

$$B(u_b + u_o, v_o) = F(v_o).$$

(5)

The displacement fields $u_b$ and $u_o$ are discretized now in a standard manner by an $h$- or a $p$-approximation

$$u_b = N_b U_b$$

$$u_o = N_o U_o$$

(6) (7)

where $N_b, N_o$ are the matrices of the shape functions of the base and the overlay mesh, respectively. The resulting linear equation system

$$\begin{bmatrix} K_{bb} & K_{bo} \\ K_{bo}^T & K_{oo} \end{bmatrix} \begin{bmatrix} U_b \\ U_o \end{bmatrix} = \begin{bmatrix} F_b \\ F_o \end{bmatrix}$$

(8)

reflects the hierarchical nature of this formulation. To solve this coupled equation system we apply a block Gauss-Seidel-iteration

$$K_{bb} U_b^{(i+1)} = F_b - K_{bo} U_o^{(i)}$$

$$K_{oo} U_o^{(i+1)} = F_o - K_{bo}^T U_b^{(i+1)}.$$ 

(9)

The advantage of this approach can be readily seen by exploring the coupling terms $K_{bo}$ and $K_{bo}^T$ together with the corresponding iterates $U_b^{(i)}$ and $U_o^{(i+1)}$, respectively. Considering e.g. $K_{bo} U_o^{(i)}$ we find that this expression can be interpreted as a load vector

$$F_{bo}^{(i)} := K_{bo} U_o^{(i)} = \int_{\Omega_1} B_b^T C B_o \, d\Omega \, U_o^{(i)} = \int_{\Omega_1} B_b^T C \varepsilon_o^{(i)} \, d\Omega$$

(10)

from negative pre-strains resulting from the displacement $U_o^{(i)}$. $B_b = LN_b$ and $B_o = LN_o$ are the $B$-matrices in the standard notation as the strain operator applied to the matrices of the shape functions of the base and the overlay mesh, respectively.

The advantage of the interpretation of (10) is that it is not necessary to compute the coupling matrices explicitly. It is therefore possible to implement the $hp$-$d$ approximation using a given finite element program as a black box, simply exchanging the corresponding pre-strains from the local to the global mesh and vice versa. The only necessary additional program module is an interpolation of strains from one mesh to the other and the possibility to incorporate pre-strains as distributed loads. Further details of the $hp$-$d$ method can be found in [4, 6].
2.2 Coupling 2D and 3D problems using the $hp$-d method

The central point of the algorithm shown in the last Section was the hierarchical splitting of the approximate solution into a local and a global part. It is now not necessary, that these two parts are computed with the same method. One could, e.g. use this idea for a coupling of a finite element to a boundary element approximation. We will yet apply this $hp$-d method here to combine two- and three-dimensional discretizations. The goal is to improve a two-dimensional approximation by a local three-dimensional Ansatz in an area, where the dimensionally reduced solution differs significantly from the exact three-dimensional model.

The algorithm for performing the combination of dimensionally reduced models with a full three-dimensional Ansatz is depicted below. It coincides with the $hp$-d method except of step 10 and 14. In step 10 the solution of the dimensionally reduced model has to be extended to a three-dimensional approximation. Considering e.g. the Reissner-Mindlin plate theory, this is done by simply applying the kinematic assumptions in order to interpolate the solution - which is related to the middle surface of the plate - to the integration points of the three-dimensional model. The projection of the three-dimensional solution to the middle surface of the plate in step 14 can be performed by evaluating the approximation at the integration points of the two-dimensional plate.

1: set $i = 0$
2: repeat
3: $i = i + 1$
4: if ($i = 1$) then
5: $F_{bo} = 0$
6: else
7: compute 'pseudo-loadvector' $F_{bo}^{(i)} = \int_{\Omega} B_{b}^{T} C \varepsilon_{b}^{(i)} \, d\Omega$
8: end if
9: solve 2D-defect equation $K_{bb} U_{b}^{(i+1)} = F_{b} - F_{bo}^{(i)}$
10: extend the solution via the model assumptions to a 3D-approximation (displacements $u_{b}^{(i+1)}$ and strains $\varepsilon_{b}^{(i+1)}$)
11: compute 'pseudo-loadvector' $F_{ob}^{(i+1)} = \int_{\Omega} B_{o}^{T} C \varepsilon_{b}^{(i+1)} \, d\Omega$
12: solve 'defect equation' $K_{oo} U_{o}^{(i+1)} = F_{o} - F_{ob}^{(i+1)}$
13: define local-global 3D-solution as $\bar{u}_{b}^{(i+1)} + u_{b}^{(i+1)}$
14: project $u_{b}^{(i+1)}, \varepsilon_{b}^{(i+1)}$ to dimensionally reduced model $u_{o}^{(i+1)}, \varepsilon_{o}^{(i+1)}$
15: check convergence
16: until (converged)

Algorithm 1: $hp$-d algorithm for coupling two- and three-dimensional models

The numerical example to be considered is a three-dimensional thick-walled plate being clamped at four sides and loaded by a pressure $T_{n} = 1000$ acting only on a small part of the upper surface (see Figure 2.2). In a first step, an analysis based on the Reissner-Mindlin
plate theory is performed by applying the $p$-version on a base mesh consisting of 108 quadrilaterals in combination with the tensor product space $S_{p}^{8,8}(\Omega_{n}^{h})$ (see Figure 4). The mesh is refined towards the boundary and at reentrant corners in order to resolve boundary layers and singularities, respectively. Following the algorithm outlined above, the two-dimensional plate solution is then extended via the Reissner-Mindlin model assumptions to a three-dimensional approximation. In the vicinity of the area where the load is applied, an overlay mesh with 17 hexahedral elements is superimposed to the base mesh. The Ansatz space $S^{p,g,d}(\Omega_{nl}^{h})$ in conjunction with the polynomial degree template

$$
\mathbf{p} = \begin{pmatrix}
    u_x & u_y & u_z \\
    7 & 7 & 8 \\
    7 & 7 & 8 \\
    4 & 4 & 5
\end{pmatrix}
$$

is chosen for the three-dimensional computation, where the defect equation including the pseudo-loadvector resulting from the two-dimensional approximation is solved. For the definition of the Ansatz space and the polynomial degree template it is referred to [3]. The local-global three-dimensional $hp$-$d$ solution is then defined as the hierarchical sum of the extended global plate approximation $\mathbf{u}_{g}^{(i+1)}$ and the local three-dimensional approximation $\mathbf{u}_{l}^{(i+1)}$. As a result of the computations consider the von Mises stress along the cut lines C-C and D-D being
plotted in Figures 5 and 6. The reference solution is based on a three-dimensional overkill approximation with a numerical error being negligibly small. From Figures 5 and 6 it is evident, that the Reissner-Mindlin approximation coincides very well with the reference solution for a wide range of the domain. Yet, in the vicinity of the area where the load is applied, the two-dimensional approximation differs significantly from the reference solution. Comparing now the $hp$-d approximation with the reference solution one observes, that the quality of the finite element solution is dramatically improved. Deviations from the reference solution are almost not visible. It is also important to notice, that only one block-Gauss-Seidel iteration step was performed in this example. Being compared to a three-dimensional approximation of the whole
problem, the additional numerical effort of the $hp$-$d$ method is very small. The outlined method to combine dimensionally reduced approximations with fully three-dimensional models could therefore be an attractive method leading to strongly improved solutions by only a small additional numerical effort.

3 Conclusions

A domain decomposition method for splitting a three-dimensional approximation into a local 3D part and a global dimensionally reduced approximation was presented. A simple block iteration procedure together with the $p$-version of the finite element method was used to obtain an efficient algorithm, which can easily be implemented without major modifications of a finite element program.

References


