Industrial Applications of Computational Mechanics
Cables and Beams

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FEM and Model Generation

Real Model
- Classification

Structural Model
- Abstraction

Mechanical Model
- Discretisation

Numerical Model
- Programming

Software Model
- Implementation

Hardware Model

(Building)
- Build

(Plates, Columns, Slabs)
- Design

(Kirchhoff/Reisner)
- Evaluation

(FEM)
- Post processing

(FEM-Program)
- Calculate

(Computer)
What is FEM?

- **FEM = Restriction = Method of Projection**
The space of possible deformations of the structure is restricted. The FEM-Solution is the shadow of the real solution into the selected solution space.

- **FEM = Method of equivalent loadings**
The real loading is replaced by a loading which is equivalent with respect to the work.

- **FEM = Method of Minimum of Energy**
FE-Program has its roots and possibilities in work and energy. Forces which do not contribute to the total work do not existent for the method.

- **FEM = Method of approximate influence functions**
An element and the mesh build with it is as precise as the influence function for a selected result may be modelled with the mesh.
FEM = Projection

Cable

\[-H w''(x) = p(x)\]

Cable Force ⇔ Bending Stiffness
Vertical Force ⇔ Shear Force
Solution for uniform load: quadr. Parabola

Cable element
Linear geometry of cable between nodes
Solution space: polygon displacements

\[w_h(x) = w_1\varphi_1(x) + w_2\varphi_2(x) + w_3\varphi_3(x)\]
FEM = Equivalent Loadings

- Nodal loads are not point loads
FEM = Equivalent Loadings

- Resolution of a mesh for loads
FEM – Loadings are more precise than engineering loads!

- Isoparametric Elements
- Drilling Degrees of Freedom
FEM = Minimum of Energy

\[ \int_0^l (V - V_h)^2 \, dx = \int_0^l H(w' - w'_h)^2 \, dx \Rightarrow \text{Minimum} \]

- Error:
  - Support (exact)
  - Loadings (Nodal loads)
  - Displacements (good)
  - Forces (constant)
- Exact Equilibrium if the forces act only at the nodes!
- A large error in the loads yields via integration a smaller error in the forces and an even smaller error in the displacements.
FEM = Method of Influence functions

- There is a simple fundamental solution for a singular load (e.g. Point force or single Moment etc.).
- Essential property of this fundamental solution $G_0(x,x_0)$ is, that all boundary conditions of the displacements are fulfilled and that there are no other singularities besides the point $x_0$ where the point load is located.
- The function $G_0(x,x_0)$ is called Greens Function and is the influence function for a displacement.

$$w = \frac{P}{2\pi G} \ln(z - z_0)$$
Integration of Greens Function

- If the point load is replaced by a differential loading (e.g. \( p(y)dy \)), the solution for any distributed loading is just an integral of this fundamental function over an area or a line.

\[
w(x) = \int_{0}^{l} G_0(x, y) p(y) dy
\]

- As Greens function is symmetric, \( G_0(y; x) = G_0(x; y) \), (Law from Maxwell), it is of no importance if we integrate along \( x \) or \( y \).
FEM = Method of Influence functions

- Influence function for the displacement $x_i$ of a cable is the polygon representing the solution for a load $P=1$ at point $i$.
- Influence function for the moment of a beam is the deformation obtained by a unit bend of size 1 at point $i$. 
FEM = Method of Influence functions

- If the FE-system is able to represent the influence line exactly, the solution will be exact.
- In all other cases, if the influence function is only approximated, we get an approximate solution.
- The difference between the real and the approximate solution may be used to estimate the error with rather high precision.
An easy principle!

- A FE-Program calculates approximate results, because it is using approximate Greens functions.
- As a mesh consisting of linear, quadratic or even cubic elements may represent only a few selected displacements, these elements will not allow the structure to deform in the required correct shape of the true Greens function.
- A FE-Program thus is not able to achieve an exact result in most cases.
An easy principle!
Other Conclusions

• For all results having their influence functions within the solution space of the FE-mesh, i.e. their Greens functions may be represented exactly, the FE solution $w_h$ is exact.
• The total result value is obtained by integrating the deformation of the influence function with the given loading $p(y)$.
• All results are obtained just with that value as if they would have been calculated with the above methodology.
• As the true Greens functions for stresses have always a singularity, it is evident that stresses within a FE program may be only exact if the singularity is reduced by the integration process of the load.
Error of forces

\[ G_h^1 = \text{FE-Näherung der EF für } N \text{ in Stabmitte} \]

![Diagram a](image1)

![Diagram b](image2)

\[ p(x) \]

Lastintensität

\[ \frac{l}{2} \quad \frac{l}{2} \]
What are Beam Elements?

- 3D Continua with Length >> width/height
- Simplification of possible deformations (Bernoulli-Hypothesis and shape of section does not change)
- Some Simplification for manual analysis (Elastic centre, principal axis, shear centre)
- Myth:
  - Beam elements are simple
  - Beam elements are exact
Still using beam elements?

- Contra Beam Elements
  - Old fashioned
  - FE-Model is more general
  - Problems for D-Regions

- Pro Beam Elements
  - Engineering Concept
  - Advantage in Computing
Other Limits for Beams
Frequency response beam / shell model
Local eigenforms: not a beam structure
Section of a beam element

- Planar section of deformation

\[ u = u_0 + \varphi_y z - \varphi_z y \]

\[ \sigma_x = E \varepsilon_x = E \frac{\partial u}{\partial x} = E \left[ \frac{\partial u}{\partial x} + \frac{\partial \varphi_y}{\partial x} z - \frac{\partial \varphi_z}{\partial x} y \right] \]
Sections

• A section is thus a substructure of the beam

\[ N = \int_A \sigma_x = EA \frac{\partial u}{\partial x} + EA_z \frac{\partial \phi_y}{\partial x} - EA_y \frac{\partial \phi_z}{\partial x} \]

\[ M_y = \int_A z \sigma_x = EA_z \frac{\partial u}{\partial x} + EA_{zz} \frac{\partial \phi_y}{\partial x} - EA_{yz} \frac{\partial \phi_z}{\partial x} \]

\[ M_z = \int_A y \sigma_x = EA_y \frac{\partial u}{\partial x} + EA_{yz} \frac{\partial \phi_y}{\partial x} - EA_{yy} \frac{\partial \phi_z}{\partial x} \]
**Differential equation system**

\[
\begin{bmatrix}
EA_x & EA_y & EA_z \\
EA_y & EI_y & EI_{yz} \\
EA_z & EI_{yz} & EI_z
\end{bmatrix}
\begin{bmatrix}
v^H_x \\
v^IV_x \\
v^IV_y
\end{bmatrix}
= \begin{bmatrix}
p_x \\
p_y \\
p_z
\end{bmatrix}
\]

- Bending with respect to principal axis, \( EI_{yz} = 0 \)
- Normal force referenced on gravity centre, \( A_y = A_z = 0 \)
- Shear forces referenced to the shear centre \( y_{sc}, z_{sc} \)
Sectional values

- Values may be calculated in advance:

\[ A = \int_{A} dA \]

\[ I_y = A_{zz} = \int_{A} z^2 dA \]

\[ I_z = A_{yy} = \int_{A} y^2 dA \]

\[ A = \sum_{i=0}^{n} \frac{1}{2} (z_{i+1} + z_i)(y_{i+1} - y_i) \]

\[ I_y = \sum_{i=0}^{n} \frac{1}{12} (z_{i+1} + z_i)(y_{i+1} - y_i)(z_{i+1}^2 + z_i^2) \]
Remark on effective width

- For some cases where the Bernoulli-Hypothesis is not really fulfilled, people introduce effective widths
  - For the sectional values i.e. stiffness
  - For the design itself
- Very difficult for biaxial bending
- The treatment of prestress loadings is prone for many discussions, as the normal force is acting on another effective section than the bending stress
A strange limit for Beam Theory

- Eccentric Moment creates Torsion
How to get the matrix of a Beam Element

- Closed solution for a prismatic beam:
  - No shear deformations
  - No warping
  - No second order effects
- Closed solution also possible for some special cases
  - Warping Torsion for prismatic beam
  - Initial stress for constant axial forces
- Closed Solution possible but not easy for some potential series of properties
Numerical Formulations

• Variational Method (FE-Method)
  • Solution space based on deformations
  • Strains calculated from deformations
  • Minimum of deformation energy

• Integration of differential equations
  • System of differential equations
  • Integration according Runge-Kutta
Integration method

- We calculate a transfer matrix either by exact or numerical integration of the D.E.:

\[
\begin{bmatrix}
    w \\
    \varphi \\
    M \\
    V
\end{bmatrix}_e
= \begin{bmatrix}
    a_{ij} & a_{ij} & a_{ij} & a_{ij} \\
    a_{ij} & a_{ij} & a_{ij} \\
    a_{ij} & a_{ij} \\
    a_{ij}
\end{bmatrix}
\begin{bmatrix}
    w \\
    \varphi \\
    M \\
    V
\end{bmatrix}_a
+ \begin{bmatrix}
    p_1 \\
    p_2 \\
    p_3 \\
    p_4
\end{bmatrix}_L
\]

- This matrix will be inverted yielding a stiffness matrix.
- This element is a real substructure capable to deal with any type of loading or structural properties.
Dealing with off diagonal Inertias

- Program knows only principal axis (?)
- Rotation of Solution into the system of principal axis
- Complete treatment of Integrals with $EI_{yz}$
- The independent principal axis of shear deformations allow only the latter complete approach.
Elastic Center (Gravity axis)

• Elastic centre is not a constructional element. Beams are aligned with outer faces.
• Haunches create bends in the centre axis
• Haunches create skewed length of beams
• Centre changes with construction stages (Cast in situ concrete)
• So what else?
Reference axis

a) 

b) 

c)
Reference axis

- Elastic centre axis with $N + V$
  - Results are directly applicable for design
- General reference axis
  - Not easy to control or understand
  - Superposition of actions
- Elastic centre with $D + T$
  - Similar to 2\textsuperscript{nd} Order Theory
  - Superposition of forces is possible
Beam with a Haunch
Variational Method

 Ansatzfunctions

 \[ v = \sum H_i W_i \]

 Strains

 \[ \varepsilon = \frac{\partial u}{\partial x} + \ldots \]

 Potential

 \[ \Pi = \int pw dx + \int E \varepsilon^2 dV \]
Deformation Shape Functions

- Eccentricity at the endpoints
  \[ u_{0i} = u_i + \varphi_{yi} \Delta z_i - \varphi_{zi} \Delta y_i \]
  \[ u_{0j} = u_j + \varphi_{yj} \Delta z_j - \varphi_{zj} \Delta y_j \]
- Interpolation \( u_0 \) linear, \( v, w, \varphi_x, \varphi_y \) cubic
- Displacement in Section
  \[ u = u_0 + \varphi_y (z - z_s) - \varphi_z (y - y_s) \]
Evaluation of Strains

• Location of elastic centre is NOT constant:

\[
\varepsilon_x = \frac{\partial}{\partial x} u = u'_o + \varphi'_y(z - z_s) - \varphi'_z(y - y_s)
\]

\[
- \varphi'_y z'_s + \varphi'_z y'_s
\]
Work of internal stress

\[ \Pi_i = \int E \varepsilon_x^2 dV = \frac{E A}{E A} \left[ u_o'^2 - 2u_o' \left[ \varphi_y z_s' - \varphi_z y_s' \right] \right] + \left[ \varphi_y z_s'^2 + \varphi_z y_s'^2 - 2\varphi_y z_s' \varphi_z y_s' \right] + \left[ E I_y \varphi_y'^2 + E I_z \varphi_z'^2 - 2E I_{yz} \varphi_y' \varphi_z' \right] \]

- Haunch creates normal force for a horizontal reference axis
- A haunch yields a variant moment for a constant axial force
- There is an additional coupling of primary and secondary bending if the inclination is not the same
A Haunched Beam

- \( p = 10 \, \text{kN/m} \)
- \( h = 100 \, \text{cm} \)
- \( h = 50 \, \text{cm} \)
- \( h = 100 \, \text{cm} \)
- \( L = 10.0 \, \text{m} \)
## Results

<table>
<thead>
<tr>
<th></th>
<th>( w[\text{mm}] )</th>
<th>( \text{Ne}[\text{kN}] )</th>
<th>( \text{Nm}[\text{kN}] )</th>
<th>( \text{Mye}[\text{kNm}] )</th>
<th>( \text{Mym}[\text{kNm}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Inclined axis</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CLOSED (1 element)</td>
<td>0.397</td>
<td>-80.50</td>
<td>-78.00</td>
<td>-73.58</td>
<td>31.91</td>
</tr>
<tr>
<td>CLOSED (8 elements)</td>
<td>0.208</td>
<td>-46.30</td>
<td>-43.80</td>
<td>-94.87</td>
<td>19.17</td>
</tr>
<tr>
<td>VAR (1 element)</td>
<td>0.172</td>
<td>-39.80</td>
<td>-37.30</td>
<td>-93.65</td>
<td>22.00</td>
</tr>
<tr>
<td>VAR (8 elements)</td>
<td>0.206</td>
<td>-45.80</td>
<td>-43.30</td>
<td>-95.02</td>
<td>19.14</td>
</tr>
<tr>
<td><strong>Horiz. Reference axis</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VAR (1 element)</td>
<td>0.168</td>
<td>-37.90</td>
<td>-37.90</td>
<td>-93.01</td>
<td>22.52</td>
</tr>
<tr>
<td>VAR (8 elements)</td>
<td>0.204</td>
<td>-44.20</td>
<td>-44.20</td>
<td>-94.85</td>
<td>19.10</td>
</tr>
</tbody>
</table>
Axial Force + Moments

-37.88

-44.19

-93.01

-94.85
Shear Deformations

• Not contained in the prerequisites
• Reduction of the bending stiffness by a comparison of deformations
• Theory Timoshenko/Marguerre separate deformation modes with a compatibility request $V = \frac{dM}{dx}$
• Inversion of the flexibility
  Most general approach
  For prismatic beam possible with a closed form
Non conforming element based on the variational method

\[ \varphi = \varphi_i \cdot (1 - \xi) + \varphi_j \cdot (\xi) + \varphi_m \cdot (4\xi (1 - \xi)) \]

\[ M = -\frac{EI}{L} \cdot \left[ (\varphi_j - \varphi_i) + 1.5 \cdot \varphi_m (8\xi - 4) \right] \]

\[ V = -GA \cdot \Theta = -GA \cdot \left[ \frac{(\varphi_i + \varphi_j)}{2} + \varphi_m - \frac{(u_j - u_i)}{L} \right] \]

- Non-conforming Ansatz for \( \Theta \) yields
  - Concise Beam Element
  - Including effects of haunches
  - Including all shear deformations
Winkler Assumption

\[ k_{ij} = \int_{0}^{L} EI \cdot \frac{d^2 N_p_i}{dx^2} \frac{d^2 N_p_j}{dx^2} + C \cdot N_p_i \cdot N_p_j \cdot dx \]

\[
K = \frac{EI}{L^3} \begin{bmatrix}
12 & -6L & -12 & -6L \\
-6L & 4L^2 & 6L & 2L^2 \\
-12 & 6L & 12 & 6L \\
-6L & 2L^2 & 6L & 4L^2 \\
\end{bmatrix} + \frac{C}{420} \begin{bmatrix}
156L & -22L^2 & 54L & -13L^2 \\
-22L^2 & 4L^3 & 13L^2 & -3L^3 \\
54L & 13L^2 & 156L & 22L^2 \\
-13L^2 & -3L^3 & 22L^2 & 4L^3 \\
\end{bmatrix}
\]
Disappointing Example

- FE-Beam element is a very powerful element, but it is a Finite Element.
- Displacements are only cubic parabolas.
- Simple span beam with uniform loading
  - cubic coefficients is zero (symmetric solution)

<table>
<thead>
<tr>
<th></th>
<th>Exact</th>
<th>1 element</th>
<th>2 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. moment</td>
<td>281.25</td>
<td>281.25</td>
<td>281.25 kNm</td>
</tr>
<tr>
<td>End rotations</td>
<td>41.147</td>
<td>41.147</td>
<td>41.147</td>
</tr>
<tr>
<td>Center deflection</td>
<td>19.2876</td>
<td>15.4301</td>
<td>19.2876 mm</td>
</tr>
</tbody>
</table>
Particular solution needed

\[ w = w_0 + w_p \]

\[ w_p \]

\[ w_0 \]
Real World (Saragossa Bridge-Pavillon)
Nearly Everything can be designed today

- Is it possible to build it?
- If we do an analysis at the total system, do we cover all details?

https://en.wikipedia.org/wiki/Sleipner_A

Sleipner A
Effects of Warping Torsion

- Petersen Stabilität page. 757 b)

\[ l = 16 \text{ m} \]

![Diagram showing a beam with a load and moment]
Effect of warping Torsion

- Bending stress \( \sigma = 8.43 \text{kN/cm}^2 \)
- 2\text{nd} order Torsional Buckling \( \sigma = 13.61 \text{kN/cm}^2 \)
- Warping stress \( \sigma = 8.28 \text{kN/cm}^2 \)

\textit{caveat:}

- Dischinger-Factor \( M_b = 69.47 \text{kNm}^2 \)
- Petersen with Formula 4 \( M_b = 38.03 \text{kNm}^2 \)
- Petersen with Formula 9 \( M_b = 55.04 \text{kNm}^2 \)
- FE-Element SOFiSTiK \( M_b = 54.77 \text{kNm}^2 \)
A Question
Look at the stresses
Two views for the stability problem

- The undeformed system is in an unstable equilibrium.
- Smallest deviations (imperfection) lead to a collapse.
- Two possible approaches:
  - Eigen value or bifurcation problem: Sudden failure from a differential deformed shape
  - Deformation problem: Non linear increase of deformations caused be the negative geometric (initial stress stiffness) Stiffness „Elasticity + Stress*Geometry“
Stability - Buckling

Equilibrium:

\[ P \cdot \Delta - C \cdot \Delta \cdot l \leq 0 \quad \Rightarrow \quad C \geq \frac{P}{l} \]
Differential equation = check list of parameters

\[
\left( EI(x) \cdot v''_z \right)^\prime\prime + D(x) \cdot \left( v''_z + v''_0 \right) + C \cdot v'_z + p_x(x) \cdot v'_z = p_z(x) 
\]

- Transverse Deformation \( v_z \) & stress free imperfection \( v_{z0} \)
- Bending stiffness \( EI(x) \)
- Longitudinal force \( D(x) \)
- Bedding in transverse direction \( C \)
- Load in longitudinal direction \( p_x(x) \)
- Load in transverse direction \( p_z(x) \)
Warping Torsion

- 7. Degree of freedom + Warping Moment $M_b$
- secondary torsional moment $M_{t2}$

\[ M_t = M_{tv} + M_{t2} = GI_T \vartheta' - EC_M \vartheta''' \]

- Hermitian Functions of 2nd Degree

\[ \Pi_{i_2} = \int_{L} EC_M \vartheta''^2 + GI_t \vartheta'^2 + \]

\[
N \left[ 2 \vartheta' y_m v_m + 2 \vartheta' y_m w_m + v_m^2 + w_m^2 + i_m \vartheta'^2 \right] + \\
M_y \left[ -2 \vartheta w_m'' + r_{My} \vartheta'^2 \right] + M_z \left[ 2 \vartheta v_m'' + r_{Mz} \vartheta'^2 \right] + \\
M_b \left[ r_{Mw} \vartheta'^2 \right] + M_t \left[ v_m' w_m'' - v_m'' w_m' \right]
\]
System behaviour

Bifurcation

2nd Order Theorie

Displacements

\( P \)
Solution methods

- Inhomogeneous Equation
  (2nd Order Theory, general nonlinear analysis)

\[
\left[ K_{lin} + K_{geo}(\sigma) \right] \cdot u = p
\]

- Homogeneous equation (Stability Eigen value)

\[
\left[ K_{lin} + \lambda \cdot K_{geo}(\sigma_{prim}) \right] \cdot X = 0
\]
Frequencies

• String of a Guitar
  • Increase of tension increases the frequency

• Compressive Members
  • Increase of compression decreases the frequency
  • For a certain compressive force the frequency will become zero
  • i.e. The structure will collapse once and will never recover (The period becomes infinity)
Asymptotic Behaviour?

Eigenvalue: 1.65
Asymptotic Behaviour?

Ultimate load: 1.08
Solution methods

• Closed Solutions for special cases (using trigonometric or hyperbolic functions)
• FE-Ansatz with Ansatz functions and a variational principle
• Numerical Integration of the differential equation for beam elements
Variational Approach for Geometric Stiffness

\[ k_{ij} = \int_0^L EI \cdot \frac{d^2 N_p_i}{dx^2} \frac{d^2 N_p_j}{dx^2} + P \cdot \frac{dN_p_i}{dx} \cdot \frac{dN_p_j}{dx} \, dx \]

\[ K = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} + \frac{P}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \]
### Same Ansatz problem with a single element for buckling Eigenvalues

<table>
<thead>
<tr>
<th>Euler case for prismat. beam</th>
<th>Theoretical</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>I (1 element)</td>
<td>3303</td>
<td>3328</td>
</tr>
<tr>
<td>I (2 elements)</td>
<td></td>
<td>3305</td>
</tr>
<tr>
<td>I (4 elements)</td>
<td></td>
<td>3303</td>
</tr>
<tr>
<td>I (8 elements)</td>
<td></td>
<td>3303</td>
</tr>
<tr>
<td>II (1 element)</td>
<td>13212</td>
<td>16065</td>
</tr>
<tr>
<td>II (2 elements)</td>
<td></td>
<td>13312</td>
</tr>
<tr>
<td>II (4 elements)</td>
<td></td>
<td>13219</td>
</tr>
<tr>
<td>II (8 elements)</td>
<td></td>
<td>13213</td>
</tr>
</tbody>
</table>
And here?
Combined Strategy

- Evaluation of the total stability with the total system and imperfections.
- As we do not model every single beam with its own imperfection and at least two elements we do local checks based on the representative beam for
  - Deformations / Buckling transverse to the structure
  - Truss elements
  - Lateral torsional buckling
  - But: Stiffness at start and end node is difficult to obtain!
Imperfection or Equivalent Loads?

\[ P = H + P \]
Quadratic Imperfection of 80 mm, $P=2000\text{kN}$
Equivalent forces according Design codes
Cantilever L = 8 m, P = 2000 kN, e_u = s_k/200

- Imperfection
  - Total deformation \(104.8 = 80 + 24.8\) mm
  - Bending moment \(= 2000 \times 0.1048 = 209.6\) kNm
  - Transversal force 0, shear force at top: 52.4 kN
- Equivalent force H = 20+20 kN + q = 5 kN
  - Load deformation \(25.1\) mm
  - Bending moment \(210.3\) kNm
  - Shear at top = \(40\times \cos(0.36) + 2000\times \sin(0.36) = 52.5\) kN
Imperfections versus equivalent loadings

- Using the equivalent forces there is a transversal force of 40 kN at the top reduced to zero at the bottom.
- Using imperfections the transverse force is 0 kN
- Shear deformations are caused by shear forces, but have been calculated with the transverse force. For columns the effect is small in general.
- Geometric non linear (GMNAI is ok)
Torsion 2nd Order Theory

- Torsional buckling load \( 1185 \text{ kN} \)
- Rotation \( 265 \text{ mrad} \)
- Primary torsional moment \( 292 \text{ kNcm} \)
- Torsional moment \( N \cdot i_p^2 \cdot \theta' \) \(-212 \text{ kNcm} \)
Stresses from 2nd order Torsion?

- No stresses like primary torsion
- No stresses like secondary torsion
- ➔ Shear stresses distributed similar to the normal stresses

\[ \tau_l = r \cdot \Theta' \cdot \sigma_l \]
\[ \sigma_x = \frac{\sigma_l}{\cos \beta} \approx \frac{\sigma_l}{1 - (r \Theta')^2} \]

\[ \tau_{yx} = 0 \]
\[ \tau_{xy} + \Delta \tau_{xy} \]
Components of drilling moments in FE-System 1st and 2nd order theory
Higher Geometric Nonlinear Effects

- horizontal cable with a length of 10.0 m, a sectional area of 0.84 cm$^2$ and a prestress of 1 kN with self weight.
  Taking into account the sagging of approx. 8 o/oo will increase the normal force by a factor of 2 and has considerable influence on deformation and Frequency:

<table>
<thead>
<tr>
<th></th>
<th>N [kN]</th>
<th>u [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without slack</td>
<td>1.0</td>
<td>83.36</td>
</tr>
<tr>
<td>With slack</td>
<td>1.9</td>
<td>43.95</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>f1</th>
<th>f2</th>
<th>f3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without slack</td>
<td>1.928</td>
<td>3.809</td>
<td>5.596</td>
</tr>
<tr>
<td>With slack</td>
<td>2.374</td>
<td>3.468</td>
<td>4.690</td>
</tr>
</tbody>
</table>

Movement up and down different
3rd order Theory

- Load excentericity will be reduced by geometric nonlinear effects and thus the bending stress is reduced by a factor of 2
- Only one participant taking part in that benchmark has recognized this effect!
Buckling Design on a representative beam

- Buckling curves
Slendernes requires Buckling length

- Useful results are only obtained by using 2\textsuperscript{nd} order Theory and imperfections.
- e.g. EN 1993 : Buckling in transverse direction is handled with a slenderness ratio $\lambda_{opt}$
- But this is not always wanted, as the number of imperfection cases may increase dramatically for complex spatial structures. And the superposition is only possible for cases with the same normal force.
- AISC and BS do not recommend to use 2\textsuperscript{nd} order theory (there are also some differences to PI-delta-analysis)!
- $\Rightarrow$ People want to do the design check on a single beam with a buckling length

$$S_K = \pi \sqrt{\frac{EI}{v_{Ki} \cdot D}}$$
Buckling length?

• It is not a geometrical size! And it is not related to any mesh density of a finite element structure!
• It is rather easy to get eigenvalues of the loading at the total system based on the geometric stiffness (there are more than one eigenvalue!)
• How to convert the global buckling factor to the local single beam?
• General assumption: If all local eigenvalues are above the global one, everything is ok.
Flag pole

Beam with 500 mm length has a $\beta = 32.66$!

**HEB 500 - S 235**

- $\sigma_N = N/A = 72.3$ Mpa
- $\sigma_M = M/W = 97.8$ Mpa

**Theory II. Order ($e_u = 8000/250$):**

\[
\sigma = \sigma_N + \sigma_M^{\text{II}} = 72.3 + 134.9 = 207.4
\]

**Centr. Buckling:** $8339$ kN, $s_k=16.331$ m

- $\lambda = 77$, $\omega = 1.50$
- $\sigma = \omega \sigma_N + 0.9 \sigma_M = 196.5$ MPa
First Eigenvalue not critical

HEB 500 - S 235
Buckling without antenna: 8687 kN
Classical from 1\textsuperscript{st} Eigenform 3019 kN
Classical from 2\textsuperscript{nd} Eigenform 8762 kN

Antenna tube 110/10
Fully fixed reference: 128 kN
Classical from 1\textsuperscript{st} Eigenform 126 kN
Classical from 2\textsuperscript{nd} Eigenform 365 kN
„Da haben wir den Salat“

- Buckling length is not a suitable design method in all cases!
- Thus it is not possible to write a program which may be used as a black box for that purpose!
- But: For any design with buckling curves we need some value!
A Buckling Tensile Member
Material Nonlinear Analysis of reinforced concrete beam

• The reinforcement is not known a priori
• Reinforcement may be staggered or not
• There is more than one possible solution even for the case with a given reinforcement
Basic Steps nonlinear

- Inner Iteration within section
  - Choose a strain distribution
  - Integration of stresses defined by the stress-strain law to forces and moments

- Corrective Residuas
  - Differences between inner / outer forces
  - Plastic Strains
  - Secant or Tangential stiffness
Strain or Residual based Evaluation?

- Strain based approach
  - Konsistent to FE-Method
  - slow convergence
  - i.g. stable method
  - Problems with hardening effects
Strain or Residual based Evaluation?

- Force based approach (NSTR SN)
  - “fast” convergence
  - Problems with saddle points
  - Problems with ultimate loadings
  - Not applicable in all occasions
Iteration Methods

- Incremental Strategy with/without Iteration
  - High computational effort
  - Unique solutions
  - However: Precision not guaranteed
  - Runge-Kutta-Method
Iteration methods

- Newton-Method
  - High computational effort
  - optimum quadratic convergence
  - Problems with limit values of Stiffness
Iteration Methods

- **Secant-Method**
  - Mean numerical effort
  - Rather fast convergence
  - No problems with limit values of stiffness
Iteration Methods

- Quasi-Newton-Method
  - Least numerical effort
  - Problems with non local behaviour
  - Crisfield + BFGS - Methods
New Stiffness or strains?

\[
\begin{bmatrix}
M_y \\
M_z
\end{bmatrix} = \begin{bmatrix}
EI_y & EI_{yz} \\
EI_{yz} & EI_z
\end{bmatrix} \cdot \begin{bmatrix}
k_y - k_{y,pl} \\
k_z - k_{z,pl}
\end{bmatrix}
\]

- 2 Equations – 5 unknowns
- Solutions:
  - Calculate diagonal stiffness only
  - Keep Stiffness, change plastic strains
  - Select tangential stiffness, add corrective strains
Torsion and Transverse Shear

- Reduction of deformation areas by some empirical method
- Real Energy equivalents
Unexpected Effects

H = 35 kN

P = 500 kN

Normalforce

+ 250 kN

- 250 kN
Linear Analysis

- Moment:
  - 210 kNm
  - 105 kNm

- Normal force:
  - +250 kN
  - -250 kN
Non linear Analysis

Moment
- 373 kNm
- 218 kNm

Normal force
- 731 kN
- 1231 kN
Simple Example

Concrete C 20
Reinforcement S 500

M = 107 kNm

1.00
55
30
60
Deformed System

\[ u_x = \int \varepsilon_m \cdot dx \]
Building Slab

Concrete C 20
Reinforc. S 500
Slab Thickness
h = 16 cm

g_k/q_k = 5.50 / 2.75 kN/m²
# Design for deflections

<table>
<thead>
<tr>
<th>Analysis according to</th>
<th>obtained deformation</th>
<th>allowed deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>uncracked</td>
<td>0.24</td>
<td>cm</td>
</tr>
<tr>
<td>fully cracked</td>
<td>1.79</td>
<td>1.0 cm</td>
</tr>
<tr>
<td>Incl. Tension stiffening</td>
<td>1.07</td>
<td>cm</td>
</tr>
</tbody>
</table>
Change support Condition

Concrete  C 20  
Reinforc.  S 500  
Slab thickness  
h = 16 cm

\[
g_k/q_k = 5.50 / 2.75 \text{ kN/m}^2
\]

quasi permanent combination = 5.50 + 0.3 \times 2.75 \text{ kN/m}^2

\[f = 0.51 \text{ cm}\]
Support of an Elastic Beam

- General assumption: support in the neutral center axis
- Real world is a support at the lower side
- The curvature induces deformations of the supports
- If the support is fixed a normal force is introduced
  \[ F = q \cdot l^2 / 8 / h \]
  reducing the sagging moment by a factor of 2!
Flowchart of a non linear Analysis

Nonlinear Analysis with $\gamma_f$ loading

Design with $\gamma_m$ material properties

Evaluate non linear Stiffness with $\gamma_s$ material properties

Accumulate the existing reinforcements
Safety factors

- Partial safety coefficients cannot be applied at an arbitrary location in a non-linear analysis. There is a significant difference if they are applied on the load $F$ or the resistance $R$!

$$E(\gamma_f \cdot F) \leq R\left(\frac{f}{\gamma_M \cdot \gamma_r}\right)$$

$$\gamma_r \cdot E(\gamma_f \cdot F) \leq R\left(\frac{f}{\gamma_M}\right) \neq E(\gamma_r \cdot \gamma_f \cdot F) \leq R\left(\frac{f}{\gamma_M}\right)$$
Safety Factors $\gamma_s$ on Stiffness?

- In general, only mean values are known.
- For stability problems a reduction is applied, sometimes.
- But for the stiffness there is no clear “on the safe side“ e.g. dynamics, settlements, thermal loadings, soil engineering.
- Thus, there is no generally accepted rule how to treat safety factors for the stiffness.
- Special treatment in Germany with a unified safety factor $\gamma_r$. 
Unique Solution?

**Equation:**

\[ R = \text{Internal Resistance} \]

**Graph:**

- **Stable equilibrium**:
  \[ \frac{dE}{dk} < \frac{dR}{dk} \]
- **Instable equilibrium**:
  \[ \frac{dE}{dk} > \frac{dR}{dk} \]

**Legend:**

- **M**: External Moment
- **k**: Curvature
- **R**: Internal Resistance

**Graph Notes:**

- **Optimum design**:
  \[ \cdot \cdot \cdot \]

**Diagram Description:**

- **Axes:**
  - Y-axis: M (External Moment)
  - X-axis: curvature k

**Graph Elements:**

- **Lines:** Solid and dashed lines represent stable and instable equilibriums, respectively.
- **Points:** Blue circles indicate points of stable and instable equilibrium.
A Slender Example

- EN 1992 5.2. (7)
  \( \theta_o = l/200, \ a_h = 0.707, \ a_m = 1.0 \)
  \( e = 14.1 \text{ mm} \)

- \( h = 240 \text{ mm} \)
  \( A_s = 11.3 \text{ cm}^2 \)
  \( \varepsilon_s = 0.0 \text{ o/oo} \)

- \( h = 230 \text{ mm} \)
  \( A_s = 63.7 \text{ cm}^2 \)
  \( \varepsilon_s = 1.56 \text{ o/oo} \)

\[ G = 1200 \text{ kN} \]
\[ Q = 350 \text{ kN} \]

- C 30/37
  \( b/h = 1000/300 \)
  \( h' = 40 \text{ mm} \)

- S 500
  \( A_s = 10 \varnothing 12 \)
  \( = 11.31 \text{ cm}^2 \)
Creep effects neglected!

- EN 1992-2004, clause 5.8.4. (4) states three conditions to be fulfilled when creep may be neglected:

  \[ \varphi_0 \leq 2 \]
  \[ \lambda \leq 75 \]
  \[ \frac{M}{N} \geq h \]

- All three are violated in this case!
Creep Effects included:

<table>
<thead>
<tr>
<th>h [mm]</th>
<th>Case A (deform)</th>
<th>Case B (complete)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>As [cm²]</td>
<td>εₙ [o/oo]</td>
</tr>
<tr>
<td>330</td>
<td>11,31</td>
<td>0,01</td>
</tr>
<tr>
<td>320</td>
<td>11,31</td>
<td>0,61</td>
</tr>
<tr>
<td>310</td>
<td>27,80</td>
<td>2,50</td>
</tr>
<tr>
<td>300</td>
<td>34,08</td>
<td>2,26</td>
</tr>
<tr>
<td>290</td>
<td>40,41</td>
<td>2,10</td>
</tr>
<tr>
<td>280</td>
<td>48,04</td>
<td>2,00</td>
</tr>
<tr>
<td>270</td>
<td>56,33</td>
<td>1,92</td>
</tr>
<tr>
<td>260</td>
<td>66,25</td>
<td>1,85</td>
</tr>
</tbody>
</table>
Stiffness depending on height

Linear Buckling requires $E = 6182$ MPa for $h=300$ mm!
Required Stiffness to prevent Buckling

As-req
Sensitive High Strength Concrete!

![Graph of Reinforcement vs. Time](image)

- **mean**
- **calc**
- **mean_40**
A cause of the unsteady behaviour

• For an uncracked section the deformations and 2nd Order effects are small, a low reinforcement is sufficient
• If a crack occurs, the stiffness drops suddenly, the limit condition is not the strength of the reinforcements but the stiffness to prevent linear buckling.
• Is that a problem for the reliability?
Conclusion / Remedies?

- There are/were provisions in some codes to prevent such strain distributions.
  - minimum excentricities including creep effects
  - minimum tensile strain (e.g. OEN)
  - maximum height of compressive zone (SNIP)
  - Higher safety factor for small strains (e.g. old DIN, ACI, AS)
  - Include Tension stiffening effects

- What should we do?
Conclusion

• Beam and Cable elements facilitate the engineering judgement
• However they are neither easy nor exact
• Beware of Systems where beam theory is not adequate