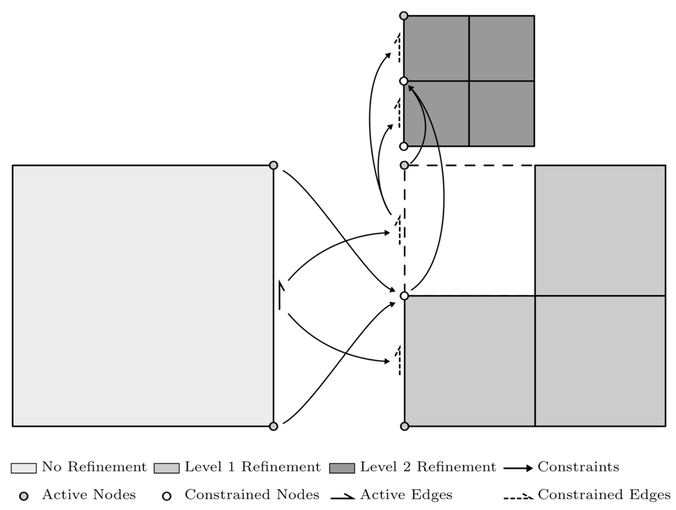
Multi-Level *hp*-Adaptivity: High-Order Mesh Adaptivity without the Difficulties of Constraining Hanging Nodes

Classical *hp*-Refinement Schemes

Idea: Coarse elements with a high error contribution are *replaced* by finer elements.

Advantage: High flexibility and possibility of automation

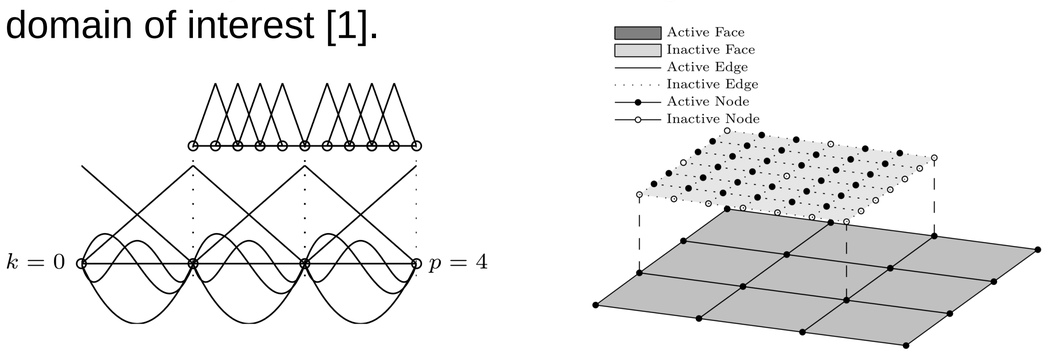
Challenge: *Hanging nodes* render the mesh irregular, as the shape functions between the interconnected elements are no longer C^0 continuous. These, therefore, require appropriate *constraining*.



The implementation of constraining is challenging, especially when multiply constrained modes have to be handled. Therefore, most *hp*-Finite Element codes restrict themselves to *1-irregular* meshes.

The *hp-d*-Refinement Scheme

Idea: Capture fine-scale solution characteristics by *superposing* a fine *h*-overlay mesh on a coarse, high-order base mesh in the domain of interest [1].



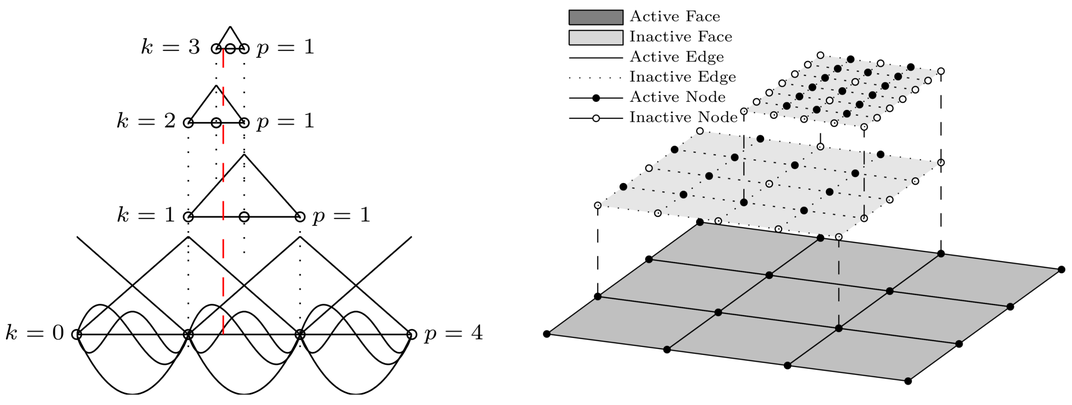
Linear independence is guaranteed by "deactivating" all direct descendants of base mesh nodes.

Compatibility is ensured by applying homogeneous Dirichlet boundary conditions on the overlay mesh.

Advantage: Hanging nodes are avoided by definition.

Hierarchical Extension

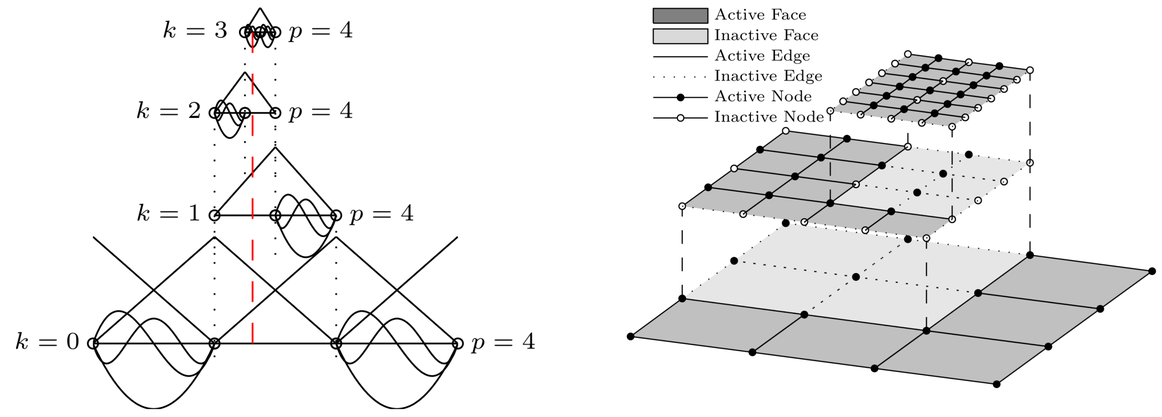
Idea: *Hierarchically* refine towards a singularity using multiple overlay meshes [2].



Advantage: Fast refinement towards singularity.

Multi-Level *hp*-Refinement

Idea: Use hierarchical, *high-oder* overlay meshes to yield the full potential of *hp*-refinement schemes [3].



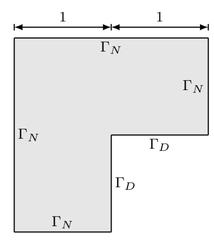
Linear independence is ensured by deactivating all topological components that have active sub-components.

Compatibility is ensured by homogeneous Dirichlet boundary conditions on the overlay meshes.

Advantage: Yield full *hp*-refinement capabilities with arbitrary hanging nodes at a comparably small implementational effort.

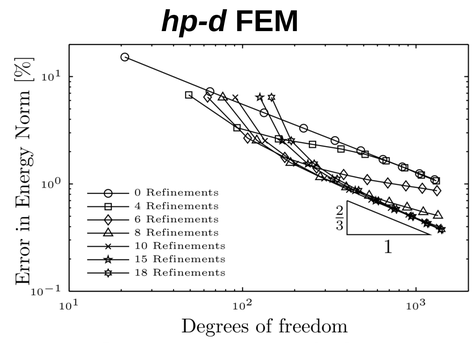
Numerical Examples

Non-Smooth Solution Benchmark

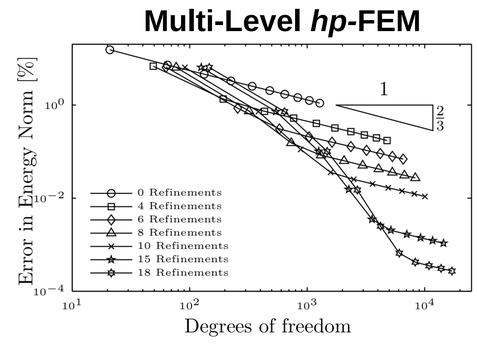


$$\Delta\phi = 0 \qquad \forall\,(r,\theta) \in \Omega$$
$$\phi(r,\theta) = 0 \qquad \forall\,(r,\theta) \in \Gamma_D$$
$$\nabla\phi(r,\theta) = \frac{2}{3} r^{-4/3} \begin{bmatrix} x\sin\left(\frac{2}{3}\theta\right) - y\cos\left(\frac{2}{3}\theta\right) \\ y\sin\left(\frac{2}{3}\theta\right) + x\cos\left(\frac{2}{3}\theta\right) \end{bmatrix} \qquad \forall\,(r,\theta) \in \Gamma_N.$$

***hp-d* FEM**



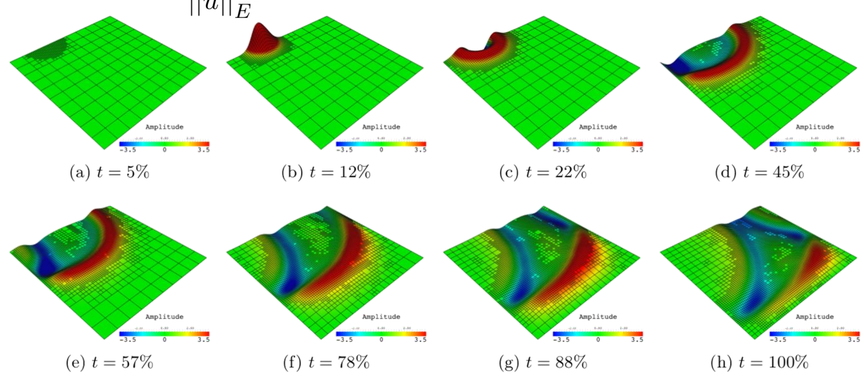
Multi-Level *hp*-FEM



Result: Exponential convergence for non-smooth problems in the pre-asymptotic range.

Automatic Multi-Level *hp*-Refinement and Coarsening

Idea: Solve the wave equation and refine elements with high energy contribution $\frac{||u||_E^e}{||u||_E^{tot}} > tol.$



References

[1] Rank, E., Adaptive remeshing and h-p domain decomposition. *Computer Methods in Applied Mechanics and Engineering*, 101(1–3):299–313, 1992

[2] Schillinger, D. The p- and B-spline versions of the geometrically nonlinear finite cell method and hierarchical refinement strategies for adaptive isogeometric and embedded domain analysis. Phd. thesis, Technische Universität München, 2012

[3] Zander, N., Bog, T., Kollmannsberger, S. Schillinger, D., Rank, E., Multi-Level hp-Adaptivity: High-Order Mesh Adaptivity without the Difficulties of Constraining Hanging Nodes, submitted to *Computational Mechanics*, 2014

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