The Hierarchical B-Spline Version of the Finite Cell Method for Geometrically Nonlinear Problems of Solid Mechanics

D. Schillinger, S. Kollmannsberger, R.-P. Mundani, E. Rank

Chair for Computation in Engineering, Department of Civil Engineering and Surveying
Technische Universität München, Arcissstr. 21, 80333 München, Germany
{schillinger,kollmannsberger,mundani,rank}@bv.tum.de

Abstract

The Finite Cell Method (FCM), which combines the fictitious domain concept with high-order $p$-FEM, permits the effective solution of problems with very complex geometry, since it circumvents the computationally expensive mesh generation and guarantees exponential convergence rates for smooth problems. The present contribution deals with the coupling of the FCM approach, which has been applied so far only to linear elasticity, with established nonlinear finite element technology. First, it is shown that the standard $p$-FEM based FCM converges poorly in a nonlinear formulation, since the presence of discontinuities leads to oscillatory solution fields. It is then demonstrated that the essential ideas of FCM, i.e. exponential convergence at virtually no meshing cost, can be achieved in the geometrically nonlinear setting, if high-order Legendre shape functions are replaced by a hierarchically enriched B-spline patch.

1 Introduction

The conventional finite element method requires the discretization of the domain of interest into a finite element mesh, whose boundaries have to coincide with the physical boundaries of the problem. While this constraint can be easily achieved for most applications in solid mechanics, difficulties arise for structures with very complex boundaries. Corresponding geometry descriptions, which are usually available in the form of CAD data or spatial voxel models, have to be transferred into finite element meshes by mesh generation algorithms that are error prone, often yield largely distorted elements and are computationally very expensive. The recently introduced Finite Cell Method (FCM) combines high-order $p$-FEM, the fictitious domain approach and adaptive integration schemes, and permits an easy rectangular mesh irrespective of the geometry [1,2]. For linear elasticity on complex domains, such as foam-like porous structures, this approach has been shown to maintain the exponential convergence property of $p$-FEM, while considerably speeding up the computation time in comparison to conventional finite elements, because costly mesh generation is omitted [1,2].

The present contribution deals with the extension of the Finite Cell Method to geometrically nonlinear problems of solid mechanics by integrating the FCM approach into the established framework of nonlinear finite element technology [3,4,5]. The paper at hand is organized as follows: First, two FCM formulations are reviewed in more detail, i.e. the standard $p$-FCM and a novel approach, which replaces high-order Legendre shape functions by hierarchical B-splines. Second, a simple one-dimensional model problem is constructed, which is suitable to demonstrate the potential of nonlinear FCM for large deformations. Third, it is shown that in a geometrically nonlinear setting, the standard $p$-FCM leads to excessive oscillations in the stress solution, which implies convergence at a very low rate. Furthermore,
it is demonstrated that in a nonlinear framework, the hierarchical B-spline FCM is able to maintain the two principal benefits of FCM, i.e. high order shape functions with exponential convergence and simple meshing without the need of an exact geometry representation. A detailed convergence analysis for the 1D test problem illustrates the advantages of the B-spline approach in comparison to \( p \)-FCM for the geometrically nonlinear case.

2 Principles of the standard \( p \)-FCM and the hierarchical B-spline FCM

In the following, the basic components of an FCM approach and the standard \( p \)-FCM are briefly reviewed. An alternative scheme is then proposed, which replaces the high-order Legendre shape functions by B-splines, which offer the possibility of local hierarchical enrichment around the location of the geometric boundaries.

2.1 Fictitious domain concept, adaptive integration and standard \( p \)-FCM

Fictitious domain methods are based on the extension of the original domain beyond its potentially complex boundaries. The resulting embedding domain, which can be conveniently meshed by a simple structured grid, consists of a physical part with true Young’s modulus \( E \), and a fictitious part, where the Young’s modulus \( E \) is multiplied by a small factor \( \alpha \ll 1 \) [1,2]. The fictitious domain concept thus exchanges the problem of complex geometry by a problem with easy geometry, but complex heterogeneous material parameters. From an algorithmic point of view, the material information is taken into account during numerical integration of the stiffness matrix at integration point level. To obtain an accurate solution, the number of Gauss points must be considerably increased near the location of the interface between physical and fictitious domains. This can be achieved in a computationally efficient way by an adaptive integration scheme, which places the bulk of the Gauss points in sub-cells along the interface (see Fig. 1b). The fictitious domain concept thus shifts the complexity from the geometry description to the numerical integration scheme.
The $p$-version of the finite element method usually uses approximation spaces that are established from the tensor product of integrated Legendre polynomials [6]. With the resulting set of high-order shape functions, convergence can be achieved by $p$-refinement (elevating the polynomial degree) rather than by $h$-refinement (choosing a denser mesh) [6]. The standard Finite Cell Method ($p$-FCM) uses shape functions of classical $p$-FEM, retaining the excellent convergence properties of the $p$-version in the linear elastic case at almost no meshing cost [1,2]. In Fig. 1, the principle of the method is illustrated for the linear elastic case by the perforated plate benchmark.

### 2.2 FCM with hierarchical B-spline shape functions

A uniform B-spline is defined as a piecewise polynomial function of degree $p$, whose $p+1$ equally spaced segments join smoothly up to a continuous differentiability of $C^p$ [7,8]. Uniform B-splines of arbitrary polynomial order $p$ can be generated efficiently by the Cox-deBoor recurrence formula [9]. A finite element basis can be constructed from a series of B-splines that are translated successively by one segment. The interfaces between the polynomial pieces of the B-spline basis are characterized by a set of coordinates in the parameter space $\xi$, which are aligned in the knot vector $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{p+2}\}$ with constant distance $\Delta \xi$ [7,9]. A knot can be repeated at the same coordinate up to $k = p + 1$ times, which reduces the differentiability at the corresponding segment interface to $C^{p-k}$. In the present finite element basis, knots are repeated $p + 1$ times at the ends of the parameter space, so that shape functions are interpolatory at the boundaries ($C^0$-continuity). Such a B-spline basis is usually denoted as a uniform B-spline patch with open knot vectors [7].

A special property of uniform B-splines is their wavelet character [10]. A B-spline with constant knot distance $\Delta \xi$ and support $(p + 1) \cdot \Delta \xi$ is linearly independent to all uniform B-splines with constant knot distance $\Delta \xi / 2^i$, $i = 1, 2, \ldots$ [8,10]. The resulting new B-spline functions with contracted support $(p + 1) \cdot \Delta \xi / 2^i$ can be characterized by their hierarchical depth $i$ and visualized in hierarchical levels as shown in Fig. 2. Due to their linear independence, the localized B-splines of lower hierarchy can be added to the uniform B-spline patch to span a new finite element basis [8]. Therefore, besides the possibility for $p$-refinement, the wavelet property of B-splines also permits a localized adaptive refinement of the approximation space.

For the construction of a hierarchical B-spline based Finite Cell Method, the B-spline basis functions are transformed from the parameter space to a regular grid of width $h$, which covers the complete embedding domain. In contrast to the standard $p$-FCM, localized hierarchical B-splines of the next hierarchical level are added to the finite element basis, if a change in material parameter $E$ is detected within a segment. The procedure can be repeated, until a sufficient hierarchical depth has been reached. This adaptive refinement can be interpreted either in the sense of $h$-refinement as in classical FEM or in the sense of local enrichment as in the extended finite element method (X-FEM). The enrichment of the finite element basis with localized functions helps to account for discontinuities in the solution fields, such as kinks in the displacements and jumps in the strains/stresses, which occur due to the change in Young’s modulus $E$. However, in contrast to the X-FEM, where the location of the parameter change has to be parameterized exactly for the construction of enrichments, hierarchical B-spline refinement does not require the information of the exact location of the parameter change.

Algorithmically, each spline segment is treated as a finite element in the classical sense, where the segment width of the lowest hierarchical B-spline of level $i$ determines the actual local element width $h/2^i$. Hierarchical B-splines do not require a specific adaptive integration concept as shown in Fig. 1b for the $p$-FCM, since the adaptive element widths automatically lead to an aggregation of Gauss points near the geometric boundaries (see Fig. 3c).
3 A simple 1D test problem at finite elastic strains

For a performance test of the suggested FCM schemes in a geometrically nonlinear setting, a 1-dimensional test problem is introduced, which consists of a rod with heterogeneous material properties as shown in Fig. 3.

3.1 Heterogeneous material properties, large deformation behaviour and self-contact

The uni-axial rod of length $L$ is divided into three sections with two material interfaces. The rod is examined under a prescribed displacement of $0.5 \cdot L$ at one end and a fixed support at the other end. The Young’s modulus $E$ of the middle part is reduced by the factor $\alpha$ according to the fictitious domain concept. The test example approximates the situation, where the $\alpha$ of the fictitious domain is exactly zero.

The outer parts of the rod can then be interpreted as two separate rods with initial distance of $\Delta X = 0.326$. The considerably larger deformation in the middle part at the beginning is an approximation of the rigid
Uniaxial rod with cross-sectional area $A = 1$, length $L = 1$ and Poisson’s ratio $\nu = 0.0$: The outer part (physical domain) is characterized by Young’s modulus $E = 1$, the middle part (fictitious domain) by $\alpha \cdot E = 0.01$.

Figure 3: Simple 1D test problem and its discretization with FCM schemes

body motion of the right rod, which would be simply translated in the direction of the left part. As soon as the stretch in the middle part approaches zero, the deformation in the outer parts suddenly increases, which can be interpreted as an approximation of the point of self-contact of the two outer parts.

3.2 Geometrically nonlinear formulation for the uni-axial rod

The physical behaviour of the suggested uni-axial rod example is assumed to be linear elastic, but large deformations in longitudinal direction are allowed [4]. With the adoption of the natural strain measure, the strains $\varepsilon$ in longitudinal direction correspond to the natural logarithm of the longitudinal stretch $\lambda_1$

$$\varepsilon = \ln(\lambda_1)$$

(1)

With a linear elastic constitutive law, the stress-strain relation follows as

$$\sigma = \frac{E}{\det(F)} \ln(\lambda_1)$$

(2)

where $\sigma$ represents the true Cauchy stress in longitudinal direction of the rod. In the present case, the deformation gradient $F$, which relates the initial to the current deformed configuration, is a diagonal tensor with the three principal stretches $\lambda_i, i = 1, 2, 3$ on its main diagonal. Since the stresses in directions orthogonal to the rod’s axis are zero, corresponding stretches $\lambda_2$ and $\lambda_3$ can be related to the longitudinal stretch $\lambda_1$ by Poisson’s ratio $\nu$ as

$$\ln(\lambda_2) = \ln(\lambda_3) = -\nu \ln(\lambda_1)$$

(3)
From Eq. (3), the determinant of the deformation gradient and the relation between initial and current rod area \(A\) and \(a\), respectively, can be obtained as

\[
\det(F) = \lambda_1^{1 - 2v} \quad (4)
\]

\[
a = \lambda_1^{-2v} A \quad (5)
\]

The stored strain energy at each position of the rod can be expressed as

\[
\Psi = \frac{E}{2} \ln(\lambda_1)^2 \quad (6)
\]

From Eq. (6), the spatial tangent modulus \(c\), associated with the consistent linearization of the stress-strain relation Eq. (2), follows as

\[
c = \frac{1}{\det(F)} \cdot \frac{\partial^2 \Psi}{\partial \ln(\lambda_1) \partial \ln(\lambda_1)} - 2\sigma = \frac{E}{\det(F)} - 2\sigma \quad (7)
\]

From this set of equations, the stress and displacement solutions can be found both analytically and numerically with the proposed FCM schemes.

### 3.3 Analytical solution of the test problem

Due to the symmetry of the rod, the longitudinal stretch in the outer parts have to be equal in each deformation step. Therefore, the test problem has the longitudinal stretches \(\lambda_{\text{out}}\) and \(\lambda_{\text{in}}\) of the outer and inner parts, respectively, as unknown variables. A first equation can be found from the statement that the averaged longitudinal stretch \(\bar{\lambda} = 0.5L\) must be equal to the sum of the unknown stretches weighted by corresponding lengths.

\[
\bar{\lambda} = \frac{1}{L} (2L_{\text{out}} \cdot \lambda_{\text{out}} + L_{\text{in}} \cdot \lambda_{\text{in}}) = 0.5 \quad (8)
\]

where \(L_{\text{in}}\) and \(L_{\text{out}}\) are the lengths of the inner and one of the outer parts, respectively. A second equation can be found from the requirement of constant stress \(\sigma\)

\[
\sigma = \frac{E}{\lambda_{\text{out}}} \ln(\lambda_{\text{out}}) = \frac{\alpha E}{\lambda_{\text{in}}} \ln(\lambda_{\text{in}}) \quad (9)
\]

where \(\det(F_{\text{out}}) = \lambda_{\text{out}}\) and \(\det(F_{\text{in}}) = \lambda_{\text{in}}\) follows from Poisson’s ratio \(\nu = 0.0\) in Eq. (4). This leads to a relation between the two unknown stretches

\[
\ln(\lambda_{\text{out}}) = \alpha \frac{\lambda_{\text{out}}}{\lambda_{\text{in}}} \ln(\lambda_{\text{in}}) \quad (10)
\]

The nonlinear equations (8) and (10) lead to the analytical reference solution \(\lambda_{\text{out}} = 0.7366, \lambda_{\text{in}} = 0.0109,\) and \(\sigma = -0.4151\). The analytical stress, displacement and deformation gradient fields are shown in Figs. 4a, 4c and 4e for 10 displacement steps between 0 and \(u_{\text{tot}}\).

### 3.4 Nonlinear finite element solution of the test problem

The nonlinear test problem can be solved by embedding the FCM schemes discussed in sections 2.1 and 2.2 into an updated Lagrange formulation [3,4,5]. The total displacement \(u_{\text{tot}}\) is divided into 10 incremental displacement steps \(u_n, n = 1, \ldots, 10\), each of which is solved by the Newton-Raphson scheme. A typical Newton iteration \(k\) consists of solving the linear system of equations

\[
K \delta u^{(k)} = -r^{(k-1)} \quad (11)
\]
where the residual $r$ is defined as the difference between internal and external forces

$$r^{(k-1)} = f^{\text{int}}(u^{(k-1)}_{n+1}) - f^{\text{ext}}_{n+1}$$

(12)

The matrix $K$ and the vector $f^{\text{int}}$ denote the tangent stiffness matrix and the internal force vector, respectively, whose components are obtained from

$$K_{ij} = \int_{\Omega} \frac{\partial N_i}{\partial x} c(u_n) \frac{\partial N_j}{\partial x} \ dx$$

(13)

$$f^{\text{int}}_i = \int_{\Omega} \frac{\partial N_i}{\partial x} \sigma(u_{n+1}) \ dx$$

(14)

where $x$ is the spatial coordinate along the rod axis and $a$ the area, both in the current configuration. In the present displacement driven example, the vector $f^{\text{ext}}$ contains only the reaction forces. After each Newton iteration, the solution $\delta u^{(k)}$ of Eq. (11) is used to update the current total displacements, until a convergence criterion is met.

4 Numerical performance of the proposed FCM schemes

For the numerical solution of the test problem, the embedding domain is discretized with two high-order elements in the case of standard $p$-FCM (see Fig. 3b), whereas B-spline patches with hierarchical enrichments as shown in Fig. 2b and 2d are used in the case of hierarchical B-spline FCM. The resulting solution fields and the convergence behaviour with respect to the analytical reference are discussed in detail in the following.

4.1 Standard $p$-FCM: Strong oscillations in the stress solution

It has been known for long that discontinuities lead to a severe performance decay of $p$-elements [6]. Here, a kink in the displacements and a jump in the strains and stresses are required, which can be reproduced only at inter-element boundaries. However, if they occur within high-order elements as in standard $p$-FCM, the continuous shape functions react by oscillating around the true discontinuous solution [6]. While in the linear case, oscillatory behaviour is restricted to the fictitious part of the domain and therefore of no harm for the solution in the physical domain [1,2], it leads to poor numerical performance, when used in a nonlinear formulation.

This is illustrated for the 1D test example in Figs. 4b, 4d and 4f, where corresponding displacements, stresses and the deformation gradient are shown. The displacement solution is unable to reconstruct the kinks at the interfaces and the stress solution is strongly oscillatory. Whereas under moderate deformations in the physical domain, the oscillations are restricted, they grow excessively under very large deformations in the fictitious domain. The occurrence of oscillations can be explained by the repeated use of the shape functions in the computation of the stiffness matrix, which are applied for calculating the Jacobian as well as the deformation gradient and enter also by its inverse via the geometric stiffness part $-2\sigma$ due to Eqs. (2) and (7). The frequent multiplication of oscillatory shape functions with each other automatically leads to a strong amplification of oscillations.

The poor numerical performance of standard $p$-FCM in the nonlinear case is further demonstrated by the relative error in strain energy [11]

$$\left(\varepsilon_x\right)_E = \sqrt{\frac{|U_{ex} - U_{FCM}|}{U_{ex}}} \cdot 100\%$$

(15)
Figure 4: Analytical reference vs. $p$-FCM solution with $p = 20$

where $U$ denotes the strain energy Eq. (6) integrated over the whole embedding domain. The corresponding error plot in Fig. 6a shows very slow convergence, if the polynomial degree $p$ is increased in the present $p$-FCM discretization.
4.2 Hierarchical B-spline FCM: \( h \)-refinement and exponential convergence

The solution fields for a cubic spline patch of polynomial order \( p = 3 \) are shown in Fig. 5. The local enrichments around each discontinuity consist of 9 hierarchical levels and 3 additional B-splines per level. In comparison to the exact reference and the standard \( p \)-FCM results, all solution fields are considerably improved. The displacements come very close to the distinct kinks at the geometric boundaries. The deformation gradient still exhibits oscillations, but they are limited to a very small area around the discontinuity. Hence, the stresses still show large oscillations, but only in the direct vicinity of the discontinuity, while the bulk of both physical and fictitious domains are free of oscillations.

Looking at the derivatives of the hierarchical B-splines (see for example Fig. 2d), one can easily understand the success of the hierarchical concept. With the addition of each hierarchical level of B-splines, the gradients of the derivatives of the lowest level steepen by factor 2. The finite element basis cannot reproduce the jump correctly, but can use the steeper gradients to approximate it increasingly well. The considerable improvement of the FCM solutions by hierarchical B-splines can also be explained by their similarity with classical \( h \)-refinement. Since oscillations only occur in elements with a discontinuity inside, a natural way to confine the oscillation phenomena is the adaptive insertion of new elements around the discontinuities. While this concept could work well in the simple 1D case, it is not realizable in higher dimensions without either creating hanging nodes or destroying the rectangular mesh structure, which lies at the heart of the FCM idea. Hierarchical B-splines, however, permit a local enrichment strategy, which limit the oscillations to the lowest applied hierarchical level and therefore has the same
Hierarchical B-Spline FCM

Figure 6: Convergence behaviour of the standard $p$-FCM and hierarchical B-spline FCM

(a) $p$-FCM vs. B-spline FCM

(b) FCM with B-splines of different $p$

Effectiveness of the hierarchical concept is demonstrated in Fig. 6a by the exponential convergence rate of the relative error in strain energy, which is not achieved by a $p$-refinement of the B-splines, but solely by the addition of more hierarchical levels to the finite element basis. The hierarchical enrichment procedure can be repeated for B-spline patches up to order $p = 9$, which yields the same convergence behaviour for each B-spline degree $p$ as shown in Fig. 6b. Further numerical studies have been carried out to test the sensitivity of the hierarchical B-spline FCM to the variation of the number of B-splines included per level and to the completeness of levels included. The results indicate that the number of 3 hierarchical B-splines per level yield an optimum performance in comparison to using only 1 or more than 3 (see Fig. 7). Furthermore, it is necessary to include all hierarchical levels from the top B-spline patch to the lowest hierarchical refinement level to obtain convergence of the FCM scheme.

Figure 7: Variation of the number $n$ of enriching B-splines per hierarchical level

(a) Example: Increase the number $n$ to 9 at each discontinuity in the cubic B-spline patch

(b) The error in strain energy with respect to different numbers $n$ reveals $n = 3$ as the optimum choice
5 Conclusions and outlook

The paper at hand shows that the standard $p$-FCM, which has been successfully applied in the linear elastic case, exhibit difficulties in the geometrically nonlinear case, since the discontinuities in material parameters at the geometric boundaries lead to large oscillations in the stress solution, which obstruct convergence. A novel B-spline based FCM approach has been proposed, which relies on hierarchical B-spline refinement around the locations of discontinuities and is able to effectively limit oscillations to the direct vicinity of discontinuities. The potential of the hierarchical B-spline FCM to maintain the advantages of the FCM approach, i.e. easy mesh generation without exact representation of geometric boundaries and exponential convergence rates, has been demonstrated for a simple nonlinear 1D test problem. Since the hierarchical B-spline FCM can be generalized to higher dimensions in a straightforward manner, the approach can be expected to constitute an effective numerical scheme for the solution of 2D and 3D nonlinear problems of solid mechanics, which is likely to achieve exponential rates of convergence by $hp$-refinement.

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References